5) Non-linear σ-models

a) Scalar fields with constraint

- $\varphi_a \varphi_a = f^2 \qquad , a = 1 \ldots N$

  constraint consistent with $SO(N)$-symmetry

  fields "live" on S^{N-1}

- $\Gamma_k = \int_x \frac{1}{2} \partial_\mu \varphi_a \partial_\mu \varphi_a$

  (no potential consistent with $SO(N)$-symmetry and constraint possible)

parametrization by orthogonal matrices:

- $\varphi_a = O_{ab} \bar{\varphi}_b \,, \qquad \bar{\varphi}_b = f \, \delta_{bN}$

  $O^T O = 1$

(arbitrary vector with length f can be obtained by rotation from $(0, 0 \ldots, f)$.)

- $O = \exp\left(i \pi_z(x) \lambda_z / f \right)$

  π_z real, $\quad \lambda_z = \lambda_z^\dagger = -\lambda_z^T = -\lambda_z^*$ generators of $SO(N)$

number of generators $\dfrac{N(N-1)}{2}$

Overcounting!

Only those generators that change $\bar{\varphi}$ should be included. Omit generators leaving $\bar{\varphi}$ invariant.

Invariance group $SO(N-1)$

Homogeneous space $SO(N)/SO(N-1)$

Number of remaining generators $\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = (N-1)\,\frac{N-2+2}{2}$

$= N-1$

$$S^{N-1} = SO(N)/SO(N-1)$$

$$\sum_z = \sum_{z=1}^{N-1}$$

$$\Gamma_2 = \int_x \frac{f^2}{2}\left(\partial_\mu O^T \partial_\mu O\right)_{NN} \qquad , \quad \varphi_a = f\, O_{aN}$$

$$\partial_\mu O = \frac{i\,\partial_\mu \pi_z}{f}\left\{ O\lambda_z + \frac{1}{2} O \left[\lambda_z , \frac{i\pi_y \lambda_y}{f}\right] + \frac{1}{6} O \left[\left[\lambda_z , \frac{i\pi_y\lambda_y}{f}\right], \frac{i\pi_w \lambda_w}{f}\right] + \dots\right.$$

$$\partial_\mu O^T = -\frac{i\,\partial_\mu \pi_z}{f}\left\{ \lambda_z O^T + \frac{1}{2}\left[\lambda_z , \frac{i\pi_y\lambda_y}{f}\right] O^T + \frac{1}{6}\left[\left[\lambda_z , \frac{i\pi_y\lambda_y}{f}\right], \frac{i\pi_w\lambda_w}{f}\right] O^T\right.$$

$$\Gamma_\ell = \int_x \frac{1}{2}\, \partial_\mu \pi_z\, \partial_\mu \pi_y \Big(\lambda_z \lambda_y$$

$$+ \frac{i}{2\rho}\, \pi_w \left[\lambda_z \lambda_y ,\, \lambda_w \right]$$

$$- \frac{1}{12\rho^2}\, \pi_w \pi_v \Big\{ 2 \left[\left[\lambda_z , \lambda_w \right], \lambda_v \right] \lambda_y + 2 \lambda_z \left[\left[\lambda_y , \lambda_w \right], \lambda_v \right]$$

$$+ 3 \left[\lambda_z , \lambda_w \right] \left[\lambda_y , \lambda_v \right] \Big\} \Big)_{NN} + \ldots$$

example $SO(3)$

$$\lambda_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$\lambda_1^2 = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 1 \end{pmatrix}, \qquad \lambda_2^2 = \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$\lambda_1 \lambda_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 \lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left[\lambda_1 , \lambda_2 \right] = i \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

generalize

$$\left(\lambda_z \lambda_y \right)_{NN} = \delta_{zy}$$

leading term in $1/\rho$ expansion

$$\Gamma_{\ell,0} = \frac{1}{2} \int_x \partial_\mu \pi_z\, \partial_\mu \pi_z$$

for $N=2$: only one generator

no interactions (no non-vanishing commutators)

„free theory"

careful: $\tilde{\pi}$ is periodic variable

$$O = \cos \frac{\tilde{\pi}}{f} + i \lambda_z \sin \frac{\tilde{\pi}}{f}$$

$\tilde{\pi} + 2\pi f$ and $\tilde{\pi}$ lead to same O, same φ

for $N > 2$: interactions

due to non-zero commutators

$$\sim \frac{1}{f^2} (\partial_\mu \pi)^2 \pi^2$$

Non-abelian non-linear σ-models are interacting theories!

coupling $g = \frac{1}{f}$

b) Non-abelian non-linear σ-models in two dimensions

$d = 2$: φ dimensionless, ρ dimensionless

g dimensionless

Mermin-Wagner theorem: continuous symmetries in $d=2$ not spontaneously broken

perturbative computation of flow of g:

$$\frac{\partial g^2}{\partial t} = - \frac{N-2}{2\pi} g^4 + \ldots$$

asymptotic freedom, similar to QCD in $d=4$

general solution

$$\frac{dg^2}{g^4} = - \frac{N-2}{2\pi} dt = - d\left(\frac{1}{g^2}\right)$$

$$\frac{1}{g^2(k)} - \frac{1}{g^2(\Lambda)} = \frac{N-2}{2\pi} \ln \frac{k}{\Lambda}$$

$$g^2(k) = \frac{g^2(\Lambda)}{1 - \frac{N-2}{2\pi} g^2(\Lambda) \ln \frac{\Lambda}{k}}$$

Kritische Skala k_c, bei der $g^2(k)$ groß wird

$$\ln \frac{\Lambda}{k_c} = \frac{2\pi}{(N-2)\, g^2(\Lambda)}$$

$$k_c = \Lambda \exp - \frac{2\pi}{(N-2)\, g^2(\Lambda)}$$

„Confinement-Skala"

Nichtlineare Beschreibung:

einfach für kleines g^2, ~~großes k~~

kompliziert für großes g^2, ~~kleines k~~

c) Relation between linear and non-linear σ-models

„Linear σ-model" $\hat{=}$ φ^4 theory

$$U = \frac{\lambda}{2}(\rho - \rho_0)^2$$

consider limit $\lambda \to \infty$

$$e^{-\int_x U} \sim \prod_x \delta(\rho(x) - \rho_0)$$

implements constraint $\frac{1}{2}\varphi_a(x)\varphi_a(x) = \rho_0$

compare non-linear σ-model $\varphi_a(x)\varphi_a(x) = f^2$

$$\Rightarrow \quad f^2 = 2\rho_0$$

Non-linear σ-model is limit of

linear σ-model with

$$\boxed{U_\Lambda = \frac{\lambda}{2}\left(\rho - \frac{f^2}{2}\right)^2, \quad \lambda \to \infty}$$

$d=2$:

$$\rho_0 = \varkappa = \frac{f^2}{2} = \frac{1}{2g^2}$$

$$g^2 = \frac{1}{2\varkappa}$$

iii) Strong coupling regime of non-linear σ-model corresponds to $\varkappa \to 0$

Symmetry restoration, Mermin-Wagner theorem

Exponentially small mass scale

$$m_R = C\Lambda \exp\left\{ - \frac{2\pi}{(N-2)g_\Lambda^2} \right\}$$

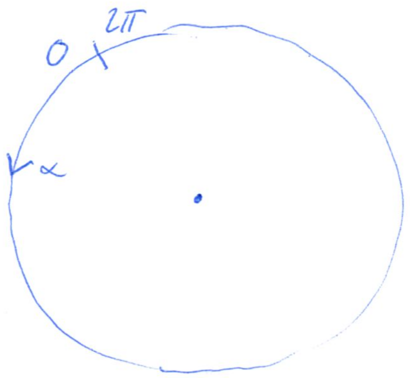
d) Kosterlitz–Thouless phase transition

$d=2, \quad N=2$

no interaction, but periodic field $\tilde{\pi}$

$\partial_t g^2 = 0 .$

Vortex



$$\tilde{\pi}(r,\alpha) = f\alpha$$

$$e^{i\tilde{\pi}\tau_2/f} = e^{i\alpha\tau_2}$$

$$= \cos\alpha + i\tau_2 \sin\alpha$$

$$\varphi = f\left(\cos\alpha + i\tau_2\sin\alpha\right)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= f\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix}$$

$$\partial_\mu\varphi_a\,\partial_\mu\varphi_a = \frac{f^2}{r^2}$$

* vortices interact !

Computation of vortex properties ;

Existence of phase transition at T_c

<u>$T < T_c$</u> (i) ~~massless~~ excitation with infinite correlation length

($\xi \to \infty$, $m_R = 0$)

(ii) critical exponent ~~η~~ depends on $T_c - T$, $\eta(T_c) = \frac{1}{4}$,

<u>$T > T_c$</u> vortex condensation

$\xi \sim \exp\left(\frac{b}{\sqrt{T-T_c}}\right)$; <u>two</u> degrees of freedom

at T_c : jump of superfluid density

„topological phase transition"

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looks like SSB with Goldstone boson,

but Mermin-Wagner theorem does not allow this

Lösung

$$\rho_0(k) = \frac{\kappa_k}{Z} \quad , \quad Z \sim \left(\frac{k}{\Lambda}\right)^{-\eta}$$

$$\rho_0(k) = \kappa_\Lambda \cdot \left(\frac{k}{\Lambda}\right)^{\eta} \quad \to 0 \text{ für } \eta > 0$$

ged

Standardbeschreibung:

Niedertemperaturphase: Wechselwirkende Vortices

Hochtemperaturphase: Kondensation der Vortices

$$\eta(\kappa_c) = \frac{1}{4}$$

Einfache Trunkierung: $\eta(\kappa_c) \approx 0.23$

lesson:

1) FRG works for large couplings

2) FRG can account for topological phase transitions