5) Non-relativistic bosons

e.g. He^4 - atoms

a) Quantum statistics in thermal equilibrium, non-zero particle number (many body problem)

$$Z = \mathrm{tr}\, \exp(-\beta(\hat{H} - \mu\hat{N})) \quad , \quad \beta = \frac{1}{T} = \frac{1}{k_B T}$$

μ: chemical potential

$\hat{H}$: Hamilton operator, $\hat{N}$: particle number operator

$$Z = \int \mathcal{D}\chi \exp(-S[\chi]) \qquad \chi(\tau, \vec{x}) = \chi(\tau+\beta, \vec{x})$$

complex field

$$S = \int_0^\beta d\tau \int_{\vec{x}} \{\chi^* \partial_\tau \chi - \chi^* \frac{\Delta}{2M}\chi - \mu \chi^*\chi + V_{int}(\chi^*\chi)\}$$

no operators anymore!

(i) free bosons: $V_{int} = 0$

momentum space: $\hat{H} = \sum_p H_p \hat{N}_p = \sum_p \frac{p^2}{2M}\hat{N}_p$, $\hat{N} = \sum_p \hat{N}_p$

$$(\hat{H} - \mu\hat{N})_p = (\frac{p^2}{2M} - \mu)\hat{N}_p$$

$$\hat{N}_p = a_p^+ a_p$$

(ii) Functional integral:

similar to quantum mechanics

* no i : $e^{-\beta \hat{H}}$ instead $e^{i\int dt \hat{H}}$

* periodicity in β : from tr

(iii) Normalization

choose $2M = 1$ $\qquad (\hbar = 1)$

τ has dimension $(\text{length})^2$

$$\int d\tau \left(\chi^* \partial_\tau \chi - \frac{\chi^* \Delta \chi}{2M} \right)$$

$$= \int \frac{d\tau}{2M} \chi^* (2M \partial_\tau - \Delta) \chi = \int d\tilde{\tau} \, \chi^* (\tilde{\partial}_\tau - \Delta) \chi$$

$\tilde{\tau} = \frac{\tau}{2M}$, dim $\tilde{\tau}$: $(\text{length})^2$

$\tilde{\mu} = 2M\mu$, dim $\tilde{\mu}$: $(\text{mass})^2 = (\text{length})^{-2}$

$$S = \int_{\tau, x} \mathcal{L}$$

$\int_{\tau,x}$ ↑ $(\text{length})^5$; $\mathcal{L}$ ↑ $(\text{mass})^5$

(iv') $T = 0$

$$\int_\tau \int_{\vec{x}} = \int d^{D+1}x = \int_x$$

D space dimensions : $d = D+1$

$\beta \to \infty$: periodicity plays no role

(v) Limit of single particle quantum mechanics

choose $T = 0$

choose μ_0 such that $\langle N \rangle = 0$

(more precisely $\langle N \rangle = 1$, but for $\mu \mp \mu_0$ $\langle N \rangle$ is a macroscopic number)

Good test for precision of many body computation!

(vi) Real fields

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_R + i\varphi_I)$$

$$\varphi^*\varphi = \frac{1}{2}(\varphi_R^2 + \varphi_I^2)$$

$$\varphi^*\partial_\tau\varphi = \frac{i}{2}(\varphi_R\partial_\tau\varphi_I - \varphi_I\partial_\tau\varphi_R)$$

off-diagonal

note $\varphi_R\partial_\tau\varphi_R = \frac{1}{2}\partial_\tau\varphi_R^2$: total derivative $\to 0$

b) dispersion relation

i) Propagator

Γ in truncation

$$\Gamma = \int_x \{ \varphi^* (\partial_\tau - \Delta - \mu) \varphi + U_{int}(\rho) \}$$

inverse propagator

$\Gamma^{(2)}$ for $\varphi_1 = \bar{\varphi}$, $\varphi_2 = 0$, $\varphi_1 \hat{=} \varphi_R$, $\varphi_2 \hat{=} \varphi_I$

$$\Gamma^{(2)} = \begin{pmatrix} -\Delta - \mu + U'_{int} + 2\rho U'', & i\partial_\tau \\ -i\partial_\tau & , -\Delta - \mu + U'_{int} \end{pmatrix}$$

$-\Delta \to p^2$, $i\partial_\tau \to -\omega_E$, $U = U_{int} - \mu\rho$

ω_E: Matsubara frequencies for $T \neq 0$

$\omega_E = 2\pi n T$ (discrete due to periodicity in β)

$$\int_{\omega_E} \hat{=} T \sum_{n=-\infty}^{\infty}$$

ii) Analytic continuation

$T=0$: Minkowski time: $\omega_E = i\omega_M$

$$\Gamma^{(2)} = G^{-1} = \begin{pmatrix} p^2 + U' + 2\rho U'', & -i\omega_M \\ i\omega_M & , p^2 + U' \end{pmatrix}$$

dispersion relation $\omega(p)$: $(\omega = \omega_M)$

Here ω_M for which G has pole

$\Rightarrow \det \Gamma^{(2)} = 0$

$$(p^2+U'+2\rho U'')(p^2+U') - \omega^2 = 0$$

$$\omega = \sqrt{(p^2+U'+2\rho U'')(p^2+U')}$$

(i) symmetric phase ; $\rho = 0$, $U'(0) = m^2$

$$\omega = p^2 + m^2$$ quadratic in p

(ii) SSB - phase, $\rho \neq 0$, $U' = 0$, $U'' = \lambda$

$$\omega = \sqrt{p^2 + 2\lambda\rho} \cdot p \quad , \quad p = \sqrt{\vec{p}^{\,2}}$$

<u>linear</u> in p for small p

<u>Bogoliubov result</u> !

no operator transformations needed

Corrections can be computed by extended truncation of Γ.

c) flow equation for effective potential

(i) derivation of flow equation

$$\partial_t U = \frac{1}{2} \int_q \mathrm{Tr}\, \partial_t R_k \left(\Gamma^{(2)} + R_k \right)^{-1}$$

$$\Gamma^{(2)} = \begin{pmatrix} \vec{q}^{\,2} + U' + 2\rho U'', & -\omega \\ \omega, & \vec{q}^{\,2} + U' \end{pmatrix} \qquad (\omega = \omega_E)$$

take $R_k(\vec{q}^{\,2})$ (independent of ω)

$$\tilde{\Gamma}^{(2)} = \Gamma^{(2)} + R_k : \qquad \vec{q}^{\,2} \to P(\vec{q}^{\,2}) = \vec{q}^{\,2} + R_k(\vec{q}^{\,2})$$

compute eigenvalues of $\tilde{\Gamma}^{(2)}$, sum over the two contributions from each eigenvalue ...

trick for simpler procedure:

$$\partial_t U = \frac{1}{2} \tilde{\partial}_t \int_q \mathrm{Tr} \ln \left(\tilde{\Gamma}^{(2)} \right), \qquad \tilde{\partial}_t = \partial_t R_k \frac{\partial}{\partial R_k}$$

$$= \frac{1}{2} \tilde{\partial}_t \ln \det \tilde{\Gamma}^{(2)}$$

$$\det \tilde{\Gamma}^{(2)} = (P + U' + 2\rho U'')(P + U') + \omega^2$$

$$\partial_t U = \frac{1}{2} \int_{\vec{q}} \int_\omega \left[(P^2 + U' + 2\rho U'')(P + U') + \omega^2 \right]^{-1}$$

$$\cdot \partial_t R_k \cdot \frac{\partial}{\partial P} (P + U' + 2\rho U'')(P + U')$$

$$\partial_t U = \int_{\vec{q}} \int_\omega \frac{\partial_t R_k \left(P + U' + \rho U'' \right)}{(P+U'+2\rho U'')(P+U') + \omega^2}$$

← (A7)

(ii) frequency integration

T = 0 : $\int_\omega = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}$

use $\int \frac{d\omega}{A+\omega^2} = \frac{1}{\sqrt{A}} \int \frac{dx}{1+x^2} = \frac{\pi}{\sqrt{A}}$ $\qquad \left(\begin{array}{l} \omega^2 = A x^2 \\ d\omega = \sqrt{A}\, dx \end{array} \right.$

$$\int_\omega \frac{1}{A+\omega^2} = \frac{1}{2\sqrt{A}}$$

$$\partial_t U = \frac{1}{2} \int_{\vec{q}} \frac{\partial_t R_k \left(P + U' + \rho U'' \right)}{\sqrt{(P+U'+2\rho U'')(P+U')}}$$

T > 0 : $\int_\omega = T \sum_{n=-\infty}^{\infty}$, $\omega = 2\pi T n$

Matsubara sum

$$\int_\omega \frac{1}{A+\omega^2} = T \sum_n \frac{1}{A + 4\pi^2 T^2 n^2} = \frac{1}{4\pi^2 T} \sum_n \frac{1}{a^2+n^2}$$

$$a^2 = \frac{A}{4\pi^2 T^2} \quad , \quad \sum_n \frac{1}{a^2+n^2} = \frac{\pi \coth(\pi a)}{a}$$

$$\int_\omega \frac{1}{A+\omega^2} = \frac{1}{2\sqrt{A}} \coth\left(\frac{\sqrt{A}}{2T} \right)$$

additional factor!

(47)

$$\partial_t U = \tilde{\partial}_t \frac{1}{2} \int_{\vec{q},\omega} \ln\left[(P+U'+2\rho U'')(P+U') + \omega^2 \right]$$

$$= \tilde{\partial}_t \, U^{(1\,loop)} \quad , \qquad \tilde{\partial}_t = \partial_t R_k \frac{\partial}{\partial R_k} = \partial_t R_k \frac{\partial}{\partial P}$$

check

$$\frac{\partial}{\partial P} \ln[\ ] = [\ ]^{-1} (P+U' + P+U'+2\rho U'')$$

$$\partial_t U = \frac{1}{2}\int_{\vec{q}} \frac{\partial_t R_k \,(P+U'+\rho U'')}{\sqrt{(p^2+U')(P+U'+2\rho U'')}} \cdot \tilde{f}(x)$$

$$\tilde{f}(x) = \coth x \qquad x = \frac{\sqrt{(P+U'+2\rho U'')(P+U')}}{2T}$$

$$= \frac{e^x+e^{-x}}{e^x-e^{-x}}$$

limit:

(1) $T \to 0, \quad x \to \infty, \quad \tilde{f}(x) \to 1$

(2) $T \to \infty, \quad x \to 0, \quad \tilde{f}(x) \to \frac{1}{x}$

$$\partial_t U = T\int_{\vec{q}} \frac{\partial_t R_k \,(P+U'+\rho U'')}{(P+U'+2\rho U'')(P+U')}$$

$$= \frac{T}{2}\int_{\vec{q}} \partial_t R_k \left(\frac{1}{P+U'} + \frac{1}{P+U'+2\rho U''} \right)$$

* D-dimensional classical statistics, quantum effects play no role
* $U_D = \frac{U}{T}$
* only $n=0$ contributes in Matsubara sum

(3) free ~~theory~~ bosons

relation to boson-occupation number in

free theory : $U' = -\mu$, $U'' = 0$

$k = 0$: $P = \vec{q}^{\,2}$

$$x = \frac{\vec{p}^{\,2} - \mu}{2T} = \frac{1}{2}\beta(E - \mu)$$

$$\frac{1}{2}\left(\tilde{f}(x) - 1\right) = \frac{1}{e^{\beta(E-\mu)} - 1} = n_B(\vec{q})$$

~~$$\partial_t U = \int_{\vec{q}} \partial_t R_k(\vec{q})\, n_B(\vec{q}) + c$$~~

~~$$c = \frac{1}{2}\int_{\vec{q}} \partial_t R_k(\vec{q})$$~~

$k \neq 0$: replace $\vec{q}^{\,2}$ by $\vec{q}^{\,2} + R_k$

We can now discuss thermodynamics of interacting bosons! ($U'' \neq 0$)

iii) <u>momentum integration</u>

Litim cutoff $\quad R_k = (k^2 - \vec{q}^{\,2})\,\Theta(k^2 - \vec{q}^{\,2})$

$$P = \begin{cases} \vec{q}^{\,2} & \text{for } \vec{q}^{\,2} > k^2 \\ k^2 & \text{for } \vec{q}^{\,2} < k^2 \end{cases}$$

$$\underline{T=0}: \quad \partial_t U = \int_{\substack{q \\ \vec{q}^{\,2} < k^2}} \frac{k^2 (k^2 + U' + \rho U'')}{\sqrt{(k^2 + U' + 2\rho U'')(k^2 + U')}} \quad ; \quad \int_{\substack{q \\ \vec{q}^{\,2} < k^2}} = \alpha_D\, k^D$$

$$\alpha_2 = \frac{1}{4\pi}, \quad \alpha_3 = \frac{1}{6\pi^2}, \quad \alpha_4 = \frac{1}{32\pi^2}$$

---

small interactions

$$\rho U'' \ll k^2 + U'$$

$$(k^2 + U' + 2\rho U'')^{-1/2} = (k^2 + U')^{-1/2} \left(1 + \frac{2\rho U''}{k^2 + U'}\right)^{-1/2}$$

$$= (k^2 + U')^{-1/2} \left(1 - \frac{\rho U''}{k^2 + U'}\right)$$

in this order

$$\partial_t U = \alpha_D\, k^{D+2} \quad : \text{ independent of } \rho\,!$$

* Interaction plays only a role in order λ^2.

iv) flow ~~in SSB~~ regime of U'

$\rho_0(k)$; neglect $U^{(3)}$ etc.

$$\partial_t U' = \alpha_D k^{D+2} \frac{\partial}{\partial\rho}\left(\frac{k^2+U'+\rho U''}{\sqrt{(k^2+U'+2\rho U'')(k^2+U')}}\right)$$

$$X = (k^2+U'+2\rho U'')(k^2+U')$$

$$\partial_t U' = \alpha_D k^{D+2}\left(2U'' X^{-1/2} - \frac{1}{2}X^{-3/2}\frac{\partial X}{\partial\rho}(k^2+U'+\rho U'')\right)$$

~~$$\frac{\partial X}{\partial\rho} = (k^2+U'+2\rho U'')U'' + (k^2+U')3U''$$~~

~~$$= U''(4k^2+4U'+2\rho U'')$$~~

omit

$$= \alpha_D k^{D+2} X^{-1/2}\left(2U'' - \frac{1}{2}\frac{\partial \ln X}{\partial\rho}(k^2+U'+\rho U'')\right)$$

$$\frac{\partial \ln X}{\partial\rho} = \frac{3U''}{k^2+U'+2\rho U''} + \frac{U''}{k^2+U'}$$

$$\partial_t U' = \alpha_D k^{D+2} X^{-1/2}\left(2U'' - \frac{3U''}{2(k^2+U'+2\rho U'')}(k^2+U'+2\rho U''-\rho U'') - \frac{U''}{2(k^2+U')}(k^2+U'+\rho U'')\right)$$

$$\partial_t U' = \alpha_D k^{D+2} X^{-1/2} U''\left(+\frac{3}{2}\frac{\rho U''}{k^2+U'+2\rho U''} - \frac{1}{2}\frac{\rho U''}{k^2+U'}\right)$$

$$\partial_t U' = \frac{1}{2}\alpha_D k^{D+2} \frac{\rho (U'')^2}{\sqrt{(\xi^2+U'+2\rho U'')(\xi^2+U')}} \left(\frac{3}{\xi^2+U'+2\rho U''} - \frac{1}{\xi^2+U'} \right)$$

<u>viii) flow in SSB-regime:</u>

flow of minimum $\rho_0(\xi)$:

$$U'(\rho_0) = 0, \quad U''(\rho_0) = \lambda, \quad \partial_t \rho_0 = -\frac{1}{\lambda}\partial_t U'(\rho_0)$$

~~neglect U''' term;~~

$$\partial_t \rho_0 = -\frac{1}{2}\alpha_D k^{D+2} \frac{\lambda \rho_0}{\sqrt{(\xi^2+2\lambda\rho_0)\xi^2}} \left(\frac{3}{\xi^2+2\lambda\rho_0} - \frac{1}{\xi^2} \right)$$

$$w = \frac{2\lambda\rho_0}{\xi^2}$$

$$\boxed{\partial_t \rho_0 = -\frac{1}{2}\alpha_D k^{D-2} \frac{\lambda\rho_0}{\sqrt{1+w}} \left(\frac{3}{1+w} - 1 \right)}$$

ix)

$\partial_t \rho_0 = 0$ for $\rho_0 = 0$ (fixed point)

for $\lambda = 0$ (free theory)

for $w = 2$ (?)

$\partial_t \rho_0 < 0$ for $w < 2$: ρ_0 increases towards IR

different from relativistic case or classical statistics in $O(N)$ models!

vi) flow in SYM regime

$$U = m^2\rho + \frac{\lambda}{2}\rho^2$$

$$U' = m^2 + \lambda\rho$$

$$\partial_t m^2 = 0 \qquad (\text{since } \rho_0 = 0)$$

(in presence of anomalous dimension: $\partial_t m^2 = \eta m^2$)

fixed point for $m^2=0$; for $m^2=0$: $\eta=0$

$$\partial_t \lambda = \frac{1}{2}\alpha_D \frac{k^{D+2}\lambda^2}{k^2+m^2}\cdot\frac{2}{k^2+m^2}$$

($\frac{\partial}{\partial\rho}$ acts on ρ in $\partial_t U'$)

$$w = \frac{m^2}{k^2}$$

$$\partial_t \lambda = \alpha_D k^{D-2}\frac{\lambda^2}{(1+w)^2}$$

fixed point for $\lambda=0$, IR attractive

upper bound for λ ; in particular $D=2$: no suppression by k^{D-2}

VIII) particle density

$$n = -\frac{\partial U}{\partial \mu}\Big|_{\rho=\rho_0,\, q=0} \qquad (\, p = -U \,)$$

SYM: $\rho_0 = 0$

SSB: $U'(\rho_0) = 0$

thermodynamics: Γ_0: Gibbs free energy (evaluated at minimum)

$$N = -T \frac{\partial}{\partial \mu} \Gamma_0 \quad , \quad \Gamma_0 = \frac{1}{T} \Omega_3 U(\rho_0)$$

lowest order: $U = -\mu \rho + U_{int}$

$$\boxed{n = \rho_0}$$

for $T=0$: $n > 0 \;\hat{=}\;$ SSB-phase

superfluidity,

n: superfluid density

in this approximation all particles in condensate at $T=0$!

omit
$$U \approx \frac{1}{2}\lambda(\rho - n)^2$$

$$U' = \lambda(\rho - n) = -\mu + U'_{int}$$

d) Bose-Einstein condensate, phase transitions

(i) quantum phase transition

(ii) $T = 0$: $n \neq 0 \hat{=} \rho_0 \neq 0$ SSB

$n \to 0$ $\rho_0 \to 0$

* Superfluidity always occurs if $n > 0$

---

quantum phase transition:

bosons are quasiparticles

quantum phase transition = phase transition at $T=0$

disordered phase : $n = 0$

no superfluid, quasi-particle number density vanishes

ordered phase $n = \rho_0 > 0$, superfluid

$n(\gamma)$, γ additional parameter

phase transition at γ_c, $n(\gamma_c) = 0$,

$n(\gamma < \gamma_c) > 0$, $n(\gamma > \gamma_c) = 0$

Critical physics at ~~[illegible]~~ phase transition ($\gamma = \gamma_c$)

Scaling solution: $m^2 = 0$ ($\rho_0 = 0$)

for $d \geq 2$: $\lambda = 0$

(for $d < 2$ $\lambda \neq 0$!)

also $\eta = 0$

mean field theory valid, at phase transition all quantities scale according to their dimension.

(ii) High temperature phase transition

modification of flow by temperature effects

given $n \neq 0$: $T < T_c$: $\rho_0 > 0$ SSB

$T > T_c$: $\rho_0 = 0$ SYM

for $\lambda \to 0$: T_c agrees with BEC - condensation of free theory

Critical exponents near T_c

Universality class of classical

D-dimensional $O(2)$-model

(Kosterlitz-Thouless for $D=2$; Heisenberg for $D=3$)

simple argument:

if phase transition is second order,
critical physics determined by long range modes
($\vec{p}^2 \to 0$); explored by $k \to 0$

for $T > 0$: the critical behavior
corresponds to $k \ll T$

$\Rightarrow$ classical statistics!

(flow equations are the same as for
classical statistics in D dimensions)

effects of quantum statistics: flow in range $k \gtrsim T$,
corresponds to $\vec{p}^2 \gtrsim (\pi T)^2$.

iii) Computation for $T > 0$

Fix renormalized couplings by $T=0$ flow

e.g. λ related to scattering length; determines microscopic parameters

Repeat computation with same microscopic parameters for $T > 0$.

iv) density in one loop order for $T > 0$

$$U = U^{(0)} + U^{(1)}$$

$$U^{(0)} = -\mu\rho + U_{int}(\rho)$$

$$U^{(1)} = \frac{1}{2}\int_{\vec{q},\omega} \ln\left[(P+U'+2\rho U'')(P+U')+\omega^2\right]$$

Check: $\partial_t U = \partial_t U^{(1)} = \frac{1}{2}\int_{\vec{q},\omega} \partial_t R_k \frac{\partial}{\partial P} \ln[\ ]$

$$\frac{P+U'+2\rho U''+P+U'}{[\ ]}$$

for $k \to 0$: $P = \vec{q}^{\,2}$

$$n = \rho_0 - \frac{\partial}{\partial\mu} U^{(1)}$$

use $U' = -\mu + U'_{int}$, $\frac{\partial}{\partial\mu} U' = -1$, $\frac{\partial}{\partial\mu}(2\rho U'') = 0$

$$n = \rho_0 + \int_{\vec{q},\omega} \frac{\vec{q}^2 + U' + \rho U''}{(q^2+U'+2\rho U'')(q^2+U') + \omega^2} \Big|_{\rho_0}$$

Free theory, n should be given by bosonic occupation numbers

neglect interactions : $2\rho U'' = 0$

SYM : $U' = -\mu \qquad (\mu \leq 0)$

$$n = \int_{\vec{q},\omega} \frac{\vec{q}^2 - \mu}{(\vec{q}^2-\mu)^2 + \omega^2}$$

ω-integration done before $\left(A = (\vec{q}^2 - \mu)^2 \right)$

$$n = \int_{\vec{q}} (\vec{q}^2-\mu) \frac{1}{2(\vec{q}^2-\mu)} \coth \frac{\vec{q}^2-\mu}{2T}$$

$$= \bar{n} + \int_{\vec{q}} f_B(\vec{q}) \quad ; \quad \bar{n} = \frac{1}{2}\int_{\vec{q}}$$

$$f_B = \frac{1}{2}\left(\coth \frac{\vec{q}^2-\mu}{2T} - 1\right) = \frac{1}{e^{\frac{\vec{q}^2-\mu}{T}} - 1}$$

f_B : ~~the~~ bosonic occupation number

$$f_B = \frac{1}{e^{\beta(E-\mu)}-1} \quad , E = \vec{q}^{\,2} \quad (\text{recall } 2M=1)$$

---

proof: $x = \frac{\vec{q}^{\,2}-\mu}{2T}$

$$\frac{1}{2}(\coth x - 1) = \frac{1}{2}\left(\frac{e^x+e^{-x}}{e^x-e^{-x}} - 1\right)$$

$$= \frac{1}{2}\left(\frac{e^{2x}+1}{e^{2x}-1} - 1\right) = \frac{1}{2}\,\frac{e^{2x}+1-e^{2x}+1}{e^{2x}-1} = \frac{1}{e^{2x}-1}$$

$\bar{n}$: divergent ~~term~~, cancels counterterm in U

$$U_{ct} = \bar{n}\mu$$

(This counterterm arises from precise translation of Hamiltonian to functional integral.)