QFT I - Problem Set 5

(9) Functional Differentiation

Sometimes (as in the next exercise) it is useful to expand functionals around known functions. The expansion of e.g. a functional $S[x(t)]$ around the path $x_0(t)$ is in close analogy to ordinary Taylor expansion given by

$$\begin{aligned} S[x(t)] &= S[x_0(t)+y(t)] \\ &= S[x_0(t)] + \int dt \frac{\delta S}{\delta x(t)}\Big|_{x_0(t)} y(t) + \frac{1}{2}\int dt dt' y(t) \frac{\delta^2 S}{\delta x(t) x(t')}\Big|_{x_0(t)} y(t') + ... \end{aligned}$$

Now suppose that S is the action of a particle in one dimension, i.e. given by

$$S[x(t)] = \int dt \mathcal{L}(\dot{x}(t), x(t)). \tag{1}$$

(a) Compute $S^{(1)} \equiv \frac{\delta S}{\delta x(t_1)}$, $S^{(2)} \equiv \frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}$ for the quadratic Lagrangian

$$\mathcal{L}(\dot{x}(t), x(t)) = c_1 \dot{x}^2(t) + c_2 x^2(t),$$

expanding around an arbitrary trajectory $x_0(t)$.

(b) Derive the Euler-Lagrange equation for general $\mathcal{L}(\dot{x}, x)$ from the action principle $\delta S/\delta x = 0$.

Hint: Functional differentiation generalizes usual differentiation. In particular, it respects general properties, such as linearity, product and chain rule. However, care has to be taken when dealing with the arguments of the functions. Useful identities are

$$\frac{\delta x(t)}{\delta x(t')} = \delta(t-t'), \quad \frac{\delta \dot{x}(t)}{\delta x(t')} = \frac{d}{dt}\delta(t-t').$$

The last identity states that functional and usual differentiation commute. To simplify expressions involving such terms, it might be useful to apply partial integration, such that d/dt does no longer act on the δ – function.

(10) Atom Propagator from the Path Integral

In exercise (6) you did already compute the propagator for a free nonrelativistic particle. Here, we would like to compute the propagator using path integrals. We restrict ourselves to one dimension. The path integral representation of the propagator is then

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q_i(t_i)}^{q_f(t_f)} \mathcal{D}q \, \exp \mathrm{i} S[q].$$

The action is given by eq. (1) with $c_1 = M/2, c_2 = 0$.

(a) By an expansion of S around the "classical path" $q_0(t)$ (*write $q(t) = q_0(t) + y(t)$; $q_0(t)$ is the solution of the Euler-Lagrange equation, i.e. q_0 solves $\delta S/\delta q = 0$*), show that the matrix element can be written as

$$\langle q_f, t_f | q_i, t_i \rangle = \exp(\mathrm{i} S[q_0]) \, \langle 0, t_f | 0, t_i \rangle.$$

with the reduced propagator

$$\langle 0, t_f | 0, t_i \rangle = \int\limits_{y(t_i)=0}^{y(t_f)=0} \mathcal{D}y \ \exp \mathrm{i} \int dt \, c_1 \, \dot{y}(t)^2 .$$

Hint: At this stage, do not solve the classical Euler-Lagrange equation explicitly. Only use the extremum property of the solution.

(b) Evaluate the reduced propagator explicitly. For this purpose, go back to the discrete version of the action,

$$\int dt c_1 \dot{y}(t)^2 = \lim_{dt \to 0, N \to \infty} dt \sum_{k=1}^{N+1} c_1 \left(\frac{y_k - y_{k-1}}{dt} \right)^2 .$$

Write this sum as a bilinear form $\kappa \, y^T A y$, using $y(t_f) = y_{N+1} = 0, y(t_i) = y_0 = 0$, with a symmetric matrix A containing only numbers. Then, evaluate the discrete path integral using the formulae of problem set 4 (*the determinant of A is* $\det A = N + 1$). Draw the continuum limit, i.e $N \to \infty$ and $dt \to 0$.

(c) Now solve the Euler-Lagrange equation for the boundary conditions $q_f(t_f) = x', q_i(t_i) = x$ (*in exercise (6), the point x' was called x and the point x was denoted x_0*). Compute the action $S[q_0]$. You should recover a expression analogous to the result of exercise (6), with the difference that we are working here in one instead of three space dimensions.

For the solution of (b) and (c), (a) is not necessary.