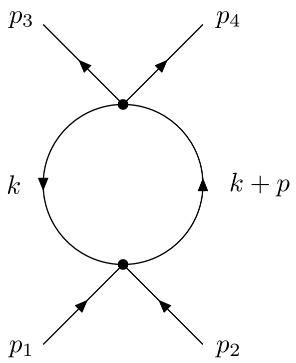
QFT II - Problem Set 4

(33) Renormalized coupling

Consider the vertex in ϕ^4 theory. We will use *Euclidian path integrals.* The vertex at arbitrary momenta p_i is obtained by computing the connected part of $\tilde{G}_4(p_1,p_2,p_3,p_4) \equiv \langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\rangle$ with external propagators amputated. To leading order, this simply gives $\langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\rangle = \lambda$.

a) At second order in λ, you get three connected diagrams. Draw them!

One of the diagrams is:



b) Use the Feynman rules

- Propagator: $\bar{G} = (p^2+m^2)^{-1}$
- Vertex: λ
- Internal loop: $\int d^4k/(2\pi)^4$ over the internal momentum k

to find an expression for the above diagram. Denote $p_1 + p_2 = p$.

c) Taking into account only the leading order λ and the diagram above, what is $\tilde{G}_4(p/2,p/2,p/2,p/2)$ to order λ^2 ? *Hint (but try first yourself): the result is on the back of this page*

d) Let us define λ_R as the coupling at vanishing momentum p:

$$\lambda_R \equiv \tilde{G}_4(0,0,0,0).$$

Without evaluating the integral, what is λ_R ?

e) Suppose that the momentum integral $\int d^4k/(2\pi)^4$ is cut-off at some UV-scale Λ and that the cut-off is low enough to ensure that the second order term contributing towards λ_R is indeed smaller than λ. Using an expansion

$$\lambda = a\lambda_R + b\lambda_R^2,$$

invert the expression for λ_R, i.e. obtain $\lambda(\lambda_R)$. *Hint: say $\lambda_R = \lambda + c\lambda^2$, then $\lambda_R = a\lambda_R + b\lambda_R^2 + ca\lambda_R^2 + \mathcal{O}(\lambda_R^3)$, which fixes a and b ...*

f) In terms of λ_R, what is the coupling at arbitrary momentum $p = 2p_1 = 2p_2$ that you obtained in (c) ? *Hint: don't evaluate the integral(s).*

g) Show that the result of (f) is finite for $\Lambda \to \infty$ by showing that the integrand becomes proportional to $1/k^2$ for $k \to \infty$.

h) Using a momentum cut-off Λ and the following simple formula for the momentum integration

$$\int_{k^2<\Lambda} d^4k = 2\pi^2 \int_0^\Lambda k^3 dk,$$

where k on the r.h.s is of course $k = \sqrt{k\cdot k} = \sqrt{k_\mu k_\nu \delta^{\mu\nu}}$, evaluate the integral in the expression for λ_R, i.e. finish the job of you started in (d).

(34) LOOP INTEGRATION OVER ARBITRARY DIMENSION

In (33), we used a simple momentum cut-off Λ to regularize our theory. There are other ways to regularize, a useful one is the so called dimensional regularization. The main idea is to work in $4+\epsilon$ dimension, perform the integral there and then let $\epsilon \to 0$. In order to do this, it's useful to know some formulae. We will derive them in the following. Consider

$$I_N = \int d^N l F(l),$$

where $F(l)$ depends only on the length of l_μ and N is an integer. Defining $L^2 = l_\mu l_\nu \delta^{\mu\nu}$, the integration meassure is

$$d^N l = L^{N-1}\mathrm{d}L\,\mathrm{d}\phi\,\sin\theta_1\mathrm{d}\theta_1\,\sin^2\theta_2\mathrm{d}\theta_2\ldots\sin^{N-2}\theta_{N-2}\mathrm{d}\theta_{N-2},$$

with $0 < L < \infty$, $0 < \phi < 2\pi$ and $0 < \theta_i < \pi$. Hence

$$I_N = 2\pi \prod_{k=1}^{N-2} \int_0^\pi \sin^k\theta_k \mathrm{d}\theta_k \int_0^\infty L^{N-1}\mathrm{d}L\,F(L). \tag{1}$$

a) Use the formula

$$\int_0^{\pi/2} (\sin t)^{2x-1}(\cos t)^{2y-1}\mathrm{d}t = \frac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and $\Gamma(1/2) = \sqrt{\pi}$ to simplify Equation (1).

b) Re-write this in terms of $x \equiv L^2$.

c) From now on, consider functions that will actually occur in loop integrals, namely

$$F(x) = (x+a^2)^{-A}.$$

Make a variable substitution $x \to y$ such that you can use the following relation for the beta function

$$B(N/2, A-N/2) = \frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(A-\frac{N}{2}\right)}{\Gamma(A)} = \int_0^\infty \mathrm{d}y\,y^{\frac{N}{2}-1}(1+y)^{-A}.$$

d) Re-express x by l^2. This yields the final expression for

$$\int \frac{\mathrm{d}^N l}{(l^2+a^2)^A}.$$

e) Let $l = l' + p$ and $b^2 = a^2 + p^2$, what relation do you get now ?

f) Derive the relation of (e) with respect to p_μ to get an expression for

$$\int \mathrm{d}^N l \frac{l_\mu}{(l^2 + 2p\cdot l + a^2)^A},$$

where $p \cdot l = p_\mu l_\nu \delta^{\mu\nu}$ as usual in euclidian space.

Solution to (33 c): $\tilde{G}_4 = \lambda - \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k+p)^2+m^2}$