QFT I - Problem Set 12

(24) Spinor and Vector indices

Show the fundamental relation for the action of the transformation of spinor and vector indices for γ matrices,

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu \tag{1}$$

with

$$\begin{aligned} S &= \exp\Big(-\frac{i}{4}\omega_{\rho\lambda}\sigma^{\rho\lambda}\Big), \\ \Lambda^\mu{}_\nu &= \exp\Big(-\frac{i}{2}\omega_{\rho\lambda}M^{\rho\lambda}\Big)^\mu{}_\nu\,. \end{aligned}$$

For this purpose, first show the relation between the generators of the Lorentz transformation in spinor space, $\sigma^{\rho\lambda}$, and the generators of boosts and rotations, $M^{\rho\lambda}$,

$$[\gamma^\mu, \sigma^{\rho\lambda}] = 2\big(M^{\rho\lambda}\big)^\mu{}_\nu \gamma^\nu \tag{2}$$

with

$$\begin{aligned} \sigma^{\rho\lambda} &= \frac{i}{2}[\gamma^\rho, \gamma^\lambda], \\ \big(M^{\rho\lambda}\big)^\mu{}_\nu &= i\big(\delta^\lambda_\nu \eta^{\rho\mu} - \eta^{\lambda\mu}\delta^\rho_\nu - \big). \end{aligned}$$

In a second step, construct the infinitesimal version of eq. (1) by using eq. (2). The desired result for finite transformations is then obtained on exponentiating, which is possible since the generators span Lie algebras.

(25) Transformation laws

Determine the behavior under Lorentz transformations of the fermion bilinears

$$i\bar\psi\mathbf{1}\psi, \quad \bar\psi\gamma^5\psi, \quad \bar\psi\gamma^\mu\psi, \quad \bar\psi\gamma^\mu\gamma^5\psi, \quad i\bar\psi\sigma^{\mu\nu}\psi\,.$$

Hint: γ^5 can be written as $\gamma^5 = \frac{i}{4!}\,\epsilon_{\mu\nu\rho\sigma}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$.

(26) The QED Lagrangian

The QED Lagrangian is given by

$$-\mathcal{L} = i\bar\psi\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\bar\psi\psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

a) A local $U(1)$ gauge transformation of the fields is given by

$$\begin{aligned} \psi \to \psi' &= e^{i\theta(x)}\psi \\ \bar\psi \to \bar\psi' &= \bar\psi e^{-i\theta(x)} \\ A_\mu \to A'_\mu &= A_\mu + \frac{1}{e}[\partial_\mu\theta(x)]. \end{aligned}$$

Show that the QED Action is invariant under such a transformation.

b) Derive the (classical) equations of motion.