QFT I - Problem Set 3

(6) Atom Propagator

Consider free nonrelativistic bosonic particles (noninteracting atoms) in a constant potential (e.g., a chemical potential μ). Their equation of motion is given by

$$\left(\mathrm{i}\partial_t + \frac{\triangle}{2M} - \mu\right)\hat{\phi}(x) = 0$$

Here, $x = (t, \boldsymbol{x})$ is a space-time coordinate. The probability amplitude for such a particle to move ("propagate") from $x_0 = (t_0, \boldsymbol{x}_0)$ to $x = (t, \boldsymbol{x})$ for $t > t_0$ is given by

$$P(x - x_0) = \theta(t - t_0)\langle x|x_0\rangle.$$

The θ - function ensures causality, i.e. the fact that the condition $t > t_0$ must hold for a nonzero amplitude in a well-posed problem.

a) Write the matrix element $\langle x|x_0\rangle$ in terms of field operators.

b) Consider the action of the differential operator $\mathcal{D} = \mathrm{i}\partial_t + \frac{\triangle}{2M} - \mu$ on the obtained expression. You should find that $P(x - x_0)$ is precisely the Green function of $\mathcal{D}$.

c) Find an explicit expression for $\tilde{P}(p)$, $p = (E, \boldsymbol{p})$ in momentum space by Fourier transform of the result in b). $\tilde{P}$ is the propagator in momentum space.

d) Knowing the propagator in momentum space transform it back to position space to obtain an explicit result for $P(x - x_0)$. In order to get the correct retarded causality perform the integration over the energy/frequency part with an infinitesimal imaginary constant $+i\epsilon$ ($\epsilon > 0$) added to the denominator of the propagator $\tilde{P}(p)$ and use the following representation of the θ - function

$$\theta(\tau) = \lim_{\epsilon \to 0} \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\omega e^{-i\omega\tau}}{\omega + i\epsilon}\,. \tag{1}$$

After integrating over the energy/frequency part, you should find

$$P(x - x_0) = -i \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x_0}) - i[\frac{p^2}{2M}+\mu](t-t_0)}\theta(t - t_0)\,. \tag{2}$$

What would change if you chose $\epsilon < 0$ in the denominator of the propagator $\tilde{P}(p)$?
Finally, perform the remaining integration over the spatial momenta.

(7) Potential Energy

This exercise might be a real eye-opener for you! We will consider the important case of two particles at positions $\boldsymbol{x}$ and $\boldsymbol{x'}$ interacting via a potential that depends on their distance $|\boldsymbol{x} - \boldsymbol{x'}|$ only, $V = V(|\boldsymbol{x}-\boldsymbol{x'}|)$. Suppose that the Hilbert space of our theory accommodates states with $0, 1, 2, \ldots$ particles at positions $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$. Let us label the states $|\boldsymbol{x}_1\boldsymbol{x}_2\ldots\rangle$. To place a particle at some position $\boldsymbol{y}$, you simply have to act with the creation operator $a^\dagger(\boldsymbol{y})|\boldsymbol{x}_1\boldsymbol{x}_2\ldots\rangle = |\boldsymbol{x}_1\boldsymbol{x}_2\ldots\boldsymbol{y}\rangle$. In addition, the usual commutation relation holds $[a(\boldsymbol{x}), a^\dagger(\boldsymbol{y})] = \delta(\boldsymbol{x} - \boldsymbol{y})$.

a) Don't look at part b)! Don't flip the page yet! How would the interaction piece of the Hamiltonian H_{int} for this case look like in terms of V, a and $a^\dagger$? *Hints: Think in terms of the familiar gravitational potential: what you want in the end is the sum of the potential energies of all pairs you can form. So clearly, in the case of a one-particle state, your interaction should return zero. And in the continuum, summing means integrating.*

b) Compute the interaction energy for a state of three particles at positions $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$, i.e.

$$\langle \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{x}_3 | H_{int} | \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{x}_3 \rangle.$$

The interaction Hamiltonian (the solution to (a)) is given by $H_{int} = \frac{1}{2} \int d\boldsymbol{x} d\boldsymbol{x}' a^\dagger(\boldsymbol{x}) a^\dagger(\boldsymbol{x}') a(\boldsymbol{x}) a(\boldsymbol{x}') V(|\boldsymbol{x} - \boldsymbol{x}'|)$.

Hints:

i) Rewrite H_{int} in terms of the number operator $n(\boldsymbol{x}) = a^\dagger(\boldsymbol{x}) a(\boldsymbol{x})$ (and $n(\boldsymbol{x}')$ of course!)

ii) Compute $n(\boldsymbol{x}) | \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{x}_3 \rangle$ to convince yourself of the action of $n(\boldsymbol{x})$ on our three-particle state.

iii) The normalization of our three particle state is such that $\langle \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{x}_3 | \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{x}_3 \rangle = [\delta(0)]^3$. If we put our field theory in a box, $\delta(0) \to$ volume of box.

iv) Finally compute the interaction energy. Whenever you get a δ-function and you have an integration left removing it, perform the integration. In other words, integrate over $\boldsymbol{x}$ and $\boldsymbol{x}'$.

c) If you like, convince yourself that you get the same answer by pulling the annihilation operators $a(\boldsymbol{x}) a(\boldsymbol{x}')$ of $H_{int} = \frac{1}{2} \int d\boldsymbol{x} d\boldsymbol{x}' a^\dagger(\boldsymbol{x}) a^\dagger(\boldsymbol{x}') a(\boldsymbol{x}) a(\boldsymbol{x}') V(|\boldsymbol{x} - \boldsymbol{x}'|)$ through the creation operators needed for our state $| \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{x}_3 \rangle = a^\dagger(\boldsymbol{x}_1) a^\dagger(\boldsymbol{x}_2) a^\dagger(\boldsymbol{x}_3) | 0 \rangle$.