QFT I - Problem Set 9

(16) A little more on problem (15): the 4-point function

If you did not yet solve problem (15), it might be a good time to do it now... Finished? Ok. You have shown in (15) that the partition function to first order in the coupling λ is

$$Z_{norm.}[j^*,j] = \Bigg\{1 + \frac{i\lambda}{2}\Big[4\int_{p_1p_2p_3} \delta(p_3-p_2)K(p_1)K(p_2)K(p_3)j(p_2)j^*(p_3)$$

$$+\int_{p_1\ldots p_4} \delta(p_1-p_2+p_3-p_4)K(p_1)K(p_2)K(p_3)K(p_4)j^*(p_1)j(p_2)j^*(p_3)j(p_4)\Big]\Bigg\} \exp\left(\int_p j^*(p)K(p)j(p)\right) \quad (1)$$

Given this expression, compute the 4-point function

$$\langle\phi^*(p_1)\phi(p_2)\phi^*(p_3)\phi(p_4)\rangle = \Big(i\frac{\delta}{\delta j(p_1)}\Big)\Big(i\frac{\delta}{\delta j^*(p_2)}\Big)\Big(i\frac{\delta}{\delta j(p_3)}\Big)\Big(i\frac{\delta}{\delta j^*(p_4)}\Big)Z_{norm.}[j^*,j]_{|j=j^*=0},$$

where $j = j^* = 0$ is of course taken after the derivatives have acted.

(17) Working with the Fermionic Functional Integral

Consider the action with an interacting fermionic field

$$\begin{aligned} S[\psi,\bar\psi] &= S_0[\psi,\bar\psi] + S_I[\psi,\bar\psi] \\ &= \int dtd^3x\Big\{\bar\psi_\alpha(t,\boldsymbol{x})\big(i\partial_t + \frac{\triangle}{2M}\big)\psi_\alpha(t,\boldsymbol{x}) + \lambda\big(\bar\psi_\alpha(t,\boldsymbol{x})\psi_\alpha(t,\boldsymbol{x})\big)^2\Big\} \end{aligned}$$

where S_0 stands for the first term which is quadratic in the fields and S_I for the remaining interacting part. Keep in mind that we sum over the repeated spin indices α.

We define the partition function as

$$Z[\bar\eta,\eta] = \int \mathcal{D}(\psi,\bar\psi)\exp\Big\{iS[\psi,\bar\psi] - i\int(\bar\eta_\alpha\psi_\alpha - \bar\psi_\alpha\eta_\alpha)\Big\},$$

where the sources η, $\bar\eta$ are Grassmann valued.

(a) Show

$$Z[\bar\eta,\eta] = \exp\big[iS_I[i\frac{\delta}{\delta\bar\eta}, i\frac{\delta}{\delta\eta}]\big]Z_0[\bar\eta,\eta]$$

with

$$Z_0[\bar\eta,\eta] = \int \mathcal{D}(\psi,\bar\psi)\exp\Big\{iS_0[\psi,\bar\psi] - i\int(\bar\eta_\alpha\psi_\alpha - \bar\psi_\alpha\eta_\alpha)\Big\}.$$

(b) Perform a Fourier transform for S_0 and the source terms. Then, evaluate Z_0 explicitly by completion of the square.

(c) Compute the free (i.e. $\lambda = 0$) fermionic two- and four-point functions.

Hint: Use the normalized generating functional $Z[\bar\eta,\eta]/\big(Z[\bar\eta,\eta]_{|\eta,\bar\eta=0}\big)$ similarly as in exercise (14).

(18) LORENTZ GROUP

a) A basis for the orthochronous Lorentz group is given by the six generators of rotations and boosts,

$$iJ_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad iJ_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad iJ_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$-iK_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad -iK_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad -iK_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Show that these generators fulfill the following Lie-algebra relations,

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k, \qquad [N_i, N_j] = i\epsilon_{ijk} N_k, \\ [J_i, K_j\,] &= i\epsilon_{ijk} K_k, \qquad [N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k^\dagger, \\ [K_i, K_j\,] &= -i\epsilon_{ijk} J_k, \qquad [N_i, N_j^\dagger] = 0, \end{aligned}$$

where on the right hand side we have introduced the basis $N_i = (J_i + iK_i)/2$.
Remark: These relations can be written in closed form as

$$\begin{aligned} M_{0i} &= -M_{i0} = K_i, \quad M_{ij} = \epsilon_{ijk} J_k, \\ [M_{\mu\nu}, M_{\rho\sigma}\,] &= -i(M_{\mu\rho}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma} + M_{\nu\sigma}\eta_{\mu\rho}) \end{aligned}$$

(with $\eta = \text{diag}(1, -1, -1, -1)$) and all Lorentz transformations can be written as $\Lambda(\omega) = \exp(i\omega^{\mu\nu} M_{\mu\nu}/2)$. The antisymmetric tensor ω contains the six real parameters of the Lorentz transformation.
b) The Pauli Lubanski vector W is given by

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}.$$

Show

$$\begin{aligned} W^\mu P_\mu &= 0, \qquad [M_{\mu\nu}, W_\sigma] = i(\eta_{\mu\sigma} W_\nu + \eta_{\nu\sigma} W_\mu), \\ [W_\mu, P_\nu\,] &= 0, \qquad [W_\mu, W_\nu] = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma. \end{aligned}$$