Quantum Field Theory I

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Chapter 1

Introduction

1.1 Particle field duality

From classical physics we know both particles and fields/waves. These are two different concepts with different characteristics. But some experiments show, that there exists a duality between both.

1. electromagnetic waves
   waves $\leftrightarrow$ photons (photoelectric effect, $E = h\nu = \hbar\omega$)
   fields are quantized, consisting of particles called photons.

2. particles (e.g. electrons) may exhibit interference phenomena, like waves. Thus, particles must be described by a wavefunction ψ. However, this has a probabilistic interpretation, it is *not* like an electromagnetic field.

The latter leads to *quantum mechanics* (QM), the former to *quantum field theory* (QFT). QM is nonrelativistic, and describes systems with fixed particle number. The quantization of the electromagnetic field requires quantum field theory, but is based on the same principles as quantum mechanics.

1.2 Short repetition of QM

QM cannot be derived from mechanics; rather, mechanics should follow from QM. But in obtaining appropriate Hamiltonians in QM, the *correspondence principle*, which substitutes quantities from mechanics by quantum mechanical operators, plays a key role.

1.2.1 Mechanics

In the Lagrangian formulation of mechanics, we substitute the equations of motion by an extremal postulate for an action functional

$$S[L] = \int_{t_0}^{t} dt L \tag{1.1}$$

of the Lagrange function $L(q_i, \dot{q}_i)$, where the q_i are the (finitely many) generalized coordinates in the specific problem. Postulating $\delta S = 0$ for variations in the $q_i(t)$ and keeping the endpoints fixed, we obtain the Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{1.2}$$

Here, $\partial L/\partial \dot{q}_i = p_i$ are the generalized canonical momenta. The Hamiltonian $H(p_i, q_i)$ is the Legendre transform of L:

$$H = \sum_i \dot{q}_i p_i - L \tag{1.3}$$

and the Hamiltonian equations in phase space follow:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1.4}$$

The Poisson bracket of two functions $f(p_i, q_i), g(p_i, q_i)$ in phase space is defined as

$$\{f(p_i, q_i), g(p_i, q_i)\}_{Poisson} = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) \tag{1.5}$$

We have

$$\{p_i, q_j\}_{Poisson} = \delta_{ij} \tag{1.6}$$

and the Hamiltonian equations can be generalized to

$$\frac{d}{dt} f(p_i, q_i) = \{H, f\}_{Poisson} \tag{1.7}$$

In case of explicitly time-dependent f we have $\dot{f}(p_i, q_i) = \{H, f\}_{Poisson} + \partial f/\partial t$. For these formal aspects see e.g. F. Scheck, "Mechanik".

Continuum mechanics can be obtained by taking the number of coordinates N to infinity, as will be seen in a specific example in QM.

1.2.2 QM States

States are described by Hilbert space (ket) vectors $|\psi\rangle \in \mathfrak{H}$ (or by the density matrix ρ; see below) with the following properties:

1. representation space: $\psi(\vec{x}) = \langle \vec{x}|\psi\rangle$ are functions in the L_2 Hilbert space $\mathfrak{H}$. These are coordinates in the $\langle\vec{x}|$-basis.

2. probabilistic interpretation:

   - $|\psi(\vec{x})|^2\, \mathrm{d}^3x$ is the probability to find the particle in the state $|\psi\rangle$ in volume element d^3x.

   - $\langle\varphi|\psi\rangle = \int \mathrm{d}^3x\ \varphi^*(\vec{x})\ \psi(\vec{x}) = \int \mathrm{d}^3p\ \varphi^*(\vec{p})\ \psi(\vec{p})$ is called the inner product of ψ and φ

   - $|\,\langle\varphi|\psi\rangle\,|^2$ is the probability to find the state $|\psi\rangle$ in $|\varphi\rangle$, and vice versa.

1.2.3 Observables

Observables are described by self-adjoint linear operators $\mathbf{A}$ in $\mathfrak{H}$: $\mathbf{A} = \mathbf{A}^\dagger$ and $\mathrm{def}(\mathbf{A}) = \mathrm{def}(\mathbf{A}^\dagger)$.

The eigenstates of $\mathbf{A}$ are orthogonal and form a complete basis, and the eigenvalues are real. The expectation value of an observable $\mathbf{A}$ in a state $|\psi\rangle$ is given by

$$\langle\mathbf{A}\rangle = \langle\psi|\mathbf{A}|\psi\rangle$$

More generally, one can introduce a *density matrix* ρ, and obtain

$$\langle\mathbf{A}\rangle = \mathrm{Tr}(\mathbf{A}\rho)$$

with

$$\rho: \quad \rho^+ = \rho, \quad \rho \geq 0, \quad \mathrm{Tr}(\rho) = 1$$

for general mixed states, and

$$\rho = P_\psi = |\psi\rangle\,\langle\psi|$$

for pure states.

1.2.4 Position and momentum

Position $\tilde{\mathbf{X}}$ and momentum $\tilde{\mathbf{P}}$ fulfill the canonical (Heisenberg) commutation relations

$$\left[\tilde{\mathbf{X}}_\mathbf{k}, \tilde{\mathbf{P}}_\mathbf{l}\right] = i\hbar\delta_{kl} \tag{1.8}$$

in accordance with the general quantization rule

$$-i\hbar \{A, B\}_{Poisson} \Rightarrow [\mathbf{A}, \mathbf{B}] \tag{1.9}$$

where the Poisson bracket for $f(x,p), g(x,p)$ is defined in eq. (1.5).

1.2.5 Hamilton operator

The correspondence principle relates mechanics to QM:

$$\text{mechanics} \left\{ \begin{array}{ccc} \vec{p} & \Rightarrow & \frac{\hbar}{i}\vec{\nabla} \\ E & \Rightarrow & i\hbar\frac{\partial}{\partial t} \\ \vec{x} & \Rightarrow & \vec{x} \end{array} \right\} \text{QM in x-space}$$

So:

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x}) \quad \Rightarrow \quad \mathbf{H} = -\frac{\hbar^2\vec{\nabla}^2}{2m} + V(\vec{x}),$$

and similarly for other operators corresponding to observables.

Time development (without measurement!)

$$|\psi(t)\rangle = \underbrace{\exp\left(-i\frac{\mathbf{H}}{\hbar}t\right)}_{=:\mathbf{U}} |\psi(0)\rangle \tag{1.10}$$

This is the representation of time development in the Schrödinger picture. The exponential function is a unitary operator ($\mathbf{U}^\dagger = \mathbf{U}^{-1}$).

In the *Schrödinger picture*, the states are time-dependent, and operators are time-independent (except when explicitly time-dependent). In contrast, in the *Heisenberg picture* the states are time-independent, and the operators are time-dependent. The expectation values are the same:

$$\langle\psi(t)|\mathbf{A}_S|\psi(t)\rangle = \Big\langle\psi(0)|\mathbf{U}^\dagger\mathbf{A}_S\mathbf{U}|\psi(0)\Big\rangle = \langle\psi(0)|\mathbf{A}_H|\psi(0)\rangle = \langle\psi_H|\mathbf{A}_H|\psi_H\rangle$$

where the Hamilton operator $\mathbf{H}$ is the same as above. The (Heisenberg) equation for time development is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_H(t) = \frac{i}{\hbar}\left[\mathbf{H}, \mathbf{A}_H(t)\right]$$

(with an additional term $\partial\mathbf{A}/\partial t$ in case of explicit time dependence in $\mathbf{A}$).

In the Heisenberg picture,

$$\psi(\vec{x},t) = \Big\langle\tilde{\mathbf{X}}_S|\psi_S(t)\Big\rangle = \Big\langle\tilde{\mathbf{X}}_S|\mathbf{U}\psi(0)\Big\rangle \stackrel{!}{=} \Big\langle\tilde{\mathbf{X}}_H|\psi_H(0)\Big\rangle$$

The actions of the operators $\tilde{\mathbf{X}}_S$ and $\tilde{\mathbf{X}}_H$ are as follows:

$$\begin{aligned} \tilde{\mathbf{X}}_S \left|x_S\right\rangle &= \vec{x}_S \left|x_S\right\rangle \\ \underbrace{\mathbf{U}^\dagger \tilde{\mathbf{X}}_S \mathbf{U}}_{\tilde{\mathbf{X}}_H} \underbrace{\mathbf{U}^\dagger \left|x_S\right\rangle}_{\left|x\right\rangle_H} &= \vec{x} \underbrace{\mathbf{U}^\dagger \left|x_S\right\rangle}_{\left|x_H\right\rangle} \end{aligned} \tag{1.11}$$

Remark

In QM, multiparticle states can be represented, in spaces like

$$\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \cdots \otimes \mathfrak{H}_N$$

which describe the whole space as a tensor product of the individual spaces of each of the N particles. This representation is used to describe e.g. atomic structure, nuclear shells, or solid state physics, but *the particle number N is always fixed!*

1.3 The need for QFT

QFT is the quantum theory of fields, the main difference to QM being the huge number of degrees of freedom ($\to \infty$). The principles, however, are the same as those of QM. There is a (multiparticle) Hilbert space, called Fock space, and a probability interpretation, all as we know them from QM. So don't worry!

The electromagnetic wave equation is the prototype of a relativistic field equation:

$$\left(\vec{\nabla}^2 - \frac{1}{c}\frac{\partial^2}{\partial t^2}\right) A(\vec{x}, t) = 0 \tag{1.12}$$

It can be solved by a wave ansatz, which leads to:

$$\left(k^2 - \frac{\omega^2}{c^2}\right) A_k e^{-i(\omega t - \vec{k}\vec{x})} = 0 \tag{1.13}$$

With $\vec{p} = \hbar\vec{k}$ and $E = \hbar\omega$ we get the *dispersion relation for photons* in the particle language:

$$\left(k^2 - \frac{\omega^2}{c^2}\right) \to p^2 - \frac{E^2}{c^2} = 0 \tag{1.14}$$

Note that the wave equation is *not* a kind of Schrödinger equation for the probability amplitude of photons. In this case it would be an equation for the probability amplitude of a *single* photon, which would lead to contradictions. Consider the physics:

- It is very easy to produce "soft" or "collinear" quanta
  "soft": $E \sim p \sim$ small
  "collinear": $\xrightarrow{\vec{p}} \overset{\vec{p}_1}{\underset{\vec{p}_2}{\rightrightarrows}}$ with $\vec{p} = \vec{p}_1 + \vec{p}_2$ and $\vec{p} \,||\, \vec{p}_1 \,||\, \vec{p}_2$

- We want to measure x with precision Δx. Assume $\Delta x \leq \lambda_{DeBroglie} = \hbar/p$. Multiplying with Δp and using the uncertainity relation we obtain $\Delta x \; \Delta p \geq \hbar \rightarrow \Delta p \geq p$. This means that new particles may be produced, because $E^2 = p^2c^2$ for massless (highly relativistic) particles.

More formally: the wave equation contains second derivatives with respect to time, which means that a probability interpretation like the one for the Schrödinger equation fails ($\sqrt{m^2c^2 + p^2}$ is nonlocal).

Remarks

- The same problems arise for *massive* relativistic particles (e.g. pion $\pi^{\pm,0}$, e): we want to measure x with precision $\Delta x \leq \frac{\hbar}{mc} = \lambda_{Compton}$ Again using $\Delta x \; \Delta p \geq \hbar$, we now obtain

  $$\Delta p \geq mc$$

  With the relativistic relation

  $$E^2 = p^2c^2 + m^2c^4 \quad \rightarrow \quad 2E\Delta E = 2p\Delta pc^2$$

  we see that

  $$\Delta E = v\Delta p \gtrsim mcv$$

  This allows particle production for $v \rightarrow c$. Thus, a particle can *not* be localized without allowing for the production of further particles.

- $\psi(\vec{x}, t)$ assumes that one can measure $\vec{x}$ arbitrarily exactly, but we have just seen that then the particle number is not conserved. QM emerges for *non-relativistic* massive particles in the limit of neglegible particle production (i.e., it is a special case of QFT). However, also for very slow massive particles there are quantum field theoretical corrections: an example is the uncertainity relation $\Delta E \Delta t \geq \hbar$, which leads to tunneling and particle production. For small times Δt a very high energy ΔE is possible, i.e. at small time scales we cannot exclude particle production.

Figure 1.1: Particle production

In short

Relativistic field equations require to be treated in the framework of QFT, where particles can be produced and annihilated. Their interpretation is different from that of the Schrödinger equation. The electromagnetic field has nothing to do with the localization probability of a single photon.

Still: The principles of QM remain true!

Other relativistic field equations are the Klein-Gordon equation and the Dirac equation. The Dirac equation, which contains only first order time derivatives, allows for a one particle interpretation in the nonrelativistic limit, although not without further ado, as will be seen later.

We will first discuss free particles, and later their interactions, almost exclusively in the context of *perturbation theory*. Particularly interesting are gauge theories (electrodynamics, chromodynamics, flavordynamics).

This might give the impression that QFT was made exclusively for fundamental theories, for elementary particle physics. However, it is also very important in statistical mechanics and solid state physics (see e.g. the role of path integrals and their relation to the partition function); and of course historically, the connection between QFT and relativity was very important!

1.4 History

1925/1926:	Heisenberg and Schrödinger develop their QM formalisms, which are proven to be equivalent
1926:	Bohm, Heisenberg and Jordan begin developing QFT
1927:	Dirac postulates his equation and explains spontaneous emission
1928:	Jordan and Wigner (anti-commutation relations, Pauli principle)
1929/1930:	Heisenberg and Pauli develop the canonical formalism
1930:	Dirac works on hole theory and antiparticles; the positron is detected
1948:	Schwinger, Feynman and Tomonaga publish on the Lamb shift; the anomalous magnetic moment e^- is explained
1949:	Dyson's work leads to a better understanding of the Feynman graph rules
1954:	Yang and Mills publish their (at the time mostly unnoticed) article on non-Abelian gauge theories
1955:	Lehmann, Szymanzik and Zimmermann (S-matrix in QFT)
1957:	Bogoliubov and Parasiuk (renormalisation)
1966:	Hepp and ...
1969:	... Zimmermann work out renormalisation to all orders

1.5 Harmonic oscillator, coherent states

1.5.1 Classical mechanics

In classical mechanics, we know the Hamiltonian of the harmonic oscillator:

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$$

with the equations of motion

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

(see Hamilton equations, sec. 1.2.1). By a canonical transformation, we obtain the *holomorphic representation*:

$$\begin{aligned} \tilde{a} &= (\sqrt{m\omega}x + i\frac{p}{\sqrt{m\omega}})/\sqrt{2} \\ \tilde{a}^* &= (\sqrt{m\omega}x + i\frac{p}{\sqrt{m\omega}})/\sqrt{2} \\ \tilde{x}, \tilde{p} &\rightarrow \tilde{a}, i\tilde{a}^* \end{aligned} \tag{1.15}$$

In terms of these new variables, the Hamiltonian becomes:

$$H = \frac{\omega}{2}\left(\tilde{a}^*\tilde{a} + \tilde{a}\tilde{a}^*\right)$$

The equations of motion are:

$$\begin{aligned} i\dot{\tilde{a}}^* &= -\frac{\partial H}{\partial \tilde{a}} = -\omega \tilde{a}^* \\ \dot{\tilde{a}} &= \frac{\partial H}{\partial (i\tilde{a}^*)} = -i\omega \tilde{a} \end{aligned} \tag{1.16}$$

These are first order differential equations, allowing us to solve for $\tilde{a}$, $\tilde{a}^*$:

$$\begin{aligned} \tilde{a}(t) &= e^{-i\omega t}\, \tilde{a}(0) \\ \tilde{a}^*(t) &= e^{i\omega t}\, \tilde{a}^*(0) \end{aligned} \tag{1.17}$$

1.5.2 Quantization

We use the quantization rule known from QM:

$$\boxed{\text{Poisson-Bracket } \rightarrow \frac{i}{\hbar} \times \text{ commutator}}$$

For example:

$$\{p, x\}_P = 1 \Rightarrow \frac{i}{\hbar}\left[\mathbf{P}, \mathbf{X}\right] = 1$$
$$\dot{f}(x,p) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial p}\dot{p} = \{H, f\}_P \Rightarrow \dot{f}(\mathbf{X}, \mathbf{P}) = \frac{i}{\hbar}\left[\mathbf{H}, f\right]$$

When we quantize the harmonic oscillator, we promote $\tilde{a}$ and $\tilde{a}^*$ to operators:

$$\tilde{a}, \tilde{a}^* \quad \Rightarrow \quad \tilde{\mathbf{a}}, \tilde{\mathbf{a}}^\dagger$$

which obey the usual commutation relation:

$$\frac{i}{\hbar}\left[i\tilde{\mathbf{a}}^\dagger, \tilde{\mathbf{a}}\right] = 1$$

In QM we will often find $\hbar$ included in $\tilde{a}$. So we can set $\hbar = 1$, or define a new $\mathbf{a}$, to make the equations look nicer:

$$\mathbf{a} := \frac{\tilde{\mathbf{a}}}{\sqrt{\hbar}} \quad \Rightarrow \quad \left[\mathbf{a}, \mathbf{a}^\dagger\right] = 1$$
$$\rightarrow \mathbf{H} = \hbar\omega \underbrace{\left(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger\right)}_{\mathbf{a}^\dagger\mathbf{a}+1} \quad \Rightarrow \quad \mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

The occupation number operator $\mathbf{N}$ is defined as

$$\mathbf{N} := \mathbf{a}^\dagger\mathbf{a}$$

and has eigenvalues

$$\mathbf{N}\,|n\rangle = n\,|n\rangle \quad n = 0, 1, ...$$

$|0\rangle$ is the ground state. In this setting $\mathbf{a}$ and $\mathbf{a}^\dagger$ are the *annihilation* and *creation* operators, and have the following actions:

$$\mathbf{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{1.18}$$
$$\mathbf{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{1.19}$$

1.5.3 Coherent states

The states $|n\rangle$ do *not* correspond to quasiclassical states. The closest approximations to classical states are called *coherent states*, which, in the Heisenberg picture, which we will use most frequently, are defined as follows:

$$|\lambda\rangle = Ne^{\lambda(t)\mathbf{a}^\dagger(t)}\,|0\rangle \tag{1.20}$$

From

$$i\hbar\frac{\partial}{\partial t}\mathbf{a}^\dagger = -\left[\mathbf{H}, \mathbf{a}^\dagger\right] = -\mathbf{a}^\dagger\hbar\omega$$

we obtain

$$\rightarrow \quad \mathbf{a}^\dagger(t) = e^{i\omega t}\mathbf{a}^\dagger(0) \rightarrow \lambda(t) = e^{-i\omega t}\lambda(0)$$

which, if plugged into the definition of a coherent state, gives:

$$|\lambda\rangle = N\sum_{n=0}^{\infty}\frac{\lambda^n\mathbf{a}^{+n}}{n!}\,|0\rangle = N\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\,|n\rangle$$

To determine the normalization N, we take the inner product of $|\lambda\rangle$ with itself:

$$\langle\lambda|\lambda\rangle = |N|^2\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!} = |N|^2e^{|\lambda|^2} \stackrel{!}{=} 1$$

$$\Rightarrow \quad N = e^{-|\lambda|^2/2} \tag{1.21}$$

If we now apply the annihilation operator $\mathbf{a}$ to such a coherent state, we obtain:

$$\mathbf{a}\,|\lambda\rangle = N\sum_{n=1}^{\infty}\frac{\lambda^n(0)\sqrt{n}}{\sqrt{n!}}|n-1\rangle = \lambda\,|\lambda\rangle \tag{1.22}$$

We can also use this as a *definition* of coherent states. From the above, it follows that the expectation values of $\mathbf{a}$ and $\mathbf{a}^\dagger$ are given by

$$\langle\lambda|\mathbf{a}|\lambda\rangle = \lambda \quad \left\langle\lambda|\mathbf{a}^\dagger|\lambda\right\rangle = \lambda^*$$

and

$$\left\langle \lambda|\mathbf{a}^\dagger\mathbf{a}|\lambda\right\rangle = |\lambda|^2$$

Analogously, we can calculate $\mathbf{a}^\dagger|\lambda\rangle$, which gives

$$\mathbf{a}^\dagger|\lambda\rangle = \frac{\mathrm{d}}{\mathrm{d}\lambda}|\lambda\rangle \tag{1.23}$$

i.e., $\mathbf{a}$ and $i\mathbf{a}^\dagger$ act on coherent states as λ and $i\frac{\mathrm{d}}{\mathrm{d}\lambda}$.

For coherent states, the sum of the variances of $\mathbf{X}$ and $\mathbf{P}$ is minimal, i.e. $|\lambda\rangle$ comes closest to classical motion.

For the variance in the occupation number N, we have:

$$\begin{aligned}
(\Delta N)^2 &= \langle\lambda|(\mathbf{N}-\langle\mathbf{N}\rangle)^2|\lambda\rangle \\
&= \langle\lambda|\mathbf{N}^2|\lambda\rangle - \langle\lambda|\mathbf{N}|\lambda\rangle^2 \\
\left\langle \lambda|\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger\mathbf{a}|\lambda\right\rangle &= |\lambda|^4+|\lambda|^2 \\
\left\langle \lambda|\mathbf{a}^\dagger\mathbf{a}|\lambda\right\rangle &= |\lambda|^2 \\
\Rightarrow (\Delta N)^2 &= |\lambda|^2 = \langle\mathbf{N}\rangle
\end{aligned}$$

$$\Rightarrow \quad \frac{\Delta N}{\langle N\rangle} = \frac{1}{\sqrt{\langle N\rangle}} \tag{1.24}$$

The relative variance goes to 0 for large N.

Remarks

- Introducing coherent states:

  In the Heisenberg picture, we have

$$\mathbf{a}|\lambda\rangle = \lambda|\lambda\rangle$$

$$|\lambda\rangle_H = e^{-|\lambda|^2/2}\underbrace{e^{\lambda(t)}}_{e^{-i\omega t}}{}^{\mathbf{a}^\dagger(t)}|0\rangle$$

  In the Schrödinger picture, this becomes:

$$\begin{aligned}
|\lambda\rangle_S &= e^{-iHt/\hbar}|\lambda\rangle_H \\
&= Ne^{\lambda(t)\mathbf{a}^\dagger(0)}|0\rangle
\end{aligned}$$

- Coherent states are not orthogonal:

$$\langle\lambda|\lambda'\rangle = e^{|\lambda-\lambda'|^2/2}e^{i\Im(\lambda^*\lambda')} \tag{1.25}$$

- The completeness relation holds for coherent states:

$$\int \frac{d\lambda d\lambda^*}{2\pi i} |\lambda\rangle \langle\lambda| = \mathbf{1} \tag{1.26}$$

We can obtain this relation by expanding an inner product $\langle n|m\rangle$ in the λ-basis and integrating:

$$\int \frac{d\lambda d\lambda^*}{2\pi i} \langle n|\lambda\rangle \langle\lambda|m\rangle = \int \frac{d\lambda d\lambda^*}{2\pi i} N N^* \frac{\lambda^n}{\sqrt{n!}} \frac{\lambda^{*m}}{\sqrt{m!}} = \delta_{nm} = \langle n|m\rangle$$

where in the second step we have used

$$\langle\lambda|m\rangle = \psi_m(\lambda^*) = N \frac{\lambda^{*m}}{\sqrt{m!}}$$

- The following normalization is often used in the literature:

$$\langle\lambda|\lambda\rangle = e^{|\lambda|^2}$$

In this case, the completeness relation changes:

$$\int \frac{d\lambda d\lambda^*}{2\pi i} e^{-\lambda^*\lambda} |\lambda\rangle \langle\lambda| = \mathbf{1}$$

- The representation of an arbitrary operator $\mathbf{A}$ in the λ-basis is as follows:

$$\begin{aligned}
\langle\lambda|\mathbf{A}f\rangle &= \frac{1}{2\pi i} \int \underbrace{\langle\lambda|\mathbf{A}|\lambda'\rangle}_{\mathbf{A}(\lambda^*,\lambda')} \langle\lambda'|f\rangle \; \mathrm{d}\lambda' \, \mathrm{d}\lambda'^* \\
\mathbf{A}(\lambda^*,\lambda') &= \langle\lambda|n\rangle \langle n|\mathbf{A}|m\rangle \langle m|\lambda'\rangle \\
&= \mathbf{A}_{mn} \frac{(\lambda^*)^n}{\sqrt{n!}} \frac{\lambda'^m}{\sqrt{m!}} e^{-|\lambda|^2/2} e^{-|\lambda'|^2/2} \quad \text{(with summing convention)} \\
\text{with} \quad \mathbf{A} &= K_{nm} \mathbf{a}^{\dagger n} \mathbf{a}^m \quad \text{(“normal ordered”)} \\
K_{nm}(\lambda^*,\lambda') &:= K_{nm} \lambda^{*n} \lambda'^m \\
\Rightarrow \quad \mathbf{A}(\lambda^*,\lambda') &= e^{-(|\lambda|^2+|\lambda'|^2)/2} K(\lambda^*,\lambda')
\end{aligned}$$

- Applying this to $\mathbf{a}$ and $\mathbf{a}^\dagger$, we plug in $K_{nm} = 0$ except for $n = 1$, $m = 0$ for $\mathbf{a}^\dagger$, and $n = 0$, $m = 1$ for $\mathbf{a}$, in which cases $K = 1$. This gives $\mathbf{a}^\dagger(\lambda^*,\lambda') = e^{-(|\lambda|^2+|\lambda'|^2)/2}\lambda^*$ and $\mathbf{a}(\lambda^*,\lambda') = e^{-(|\lambda|^2+|\lambda'|^2)/2}\lambda'$

- Canonical transformations in phase space are *symplectic* (see e.g. F. Scheck, "Mechanik"). A general phase space vector

$$\vec{z}_{Ph} = \begin{pmatrix} \vec{x} \\ \vec{p} \end{pmatrix} \quad \begin{matrix} x_1, \ldots, x_N = z_1, \ldots, z_N \\ p_1, \ldots, p_N = z_{N+1}, \ldots, z_{2N} \end{matrix}$$

has the following Hamilton equation:

$$\frac{d\vec{z}_{Ph}}{dt} = J \frac{\partial H}{\partial \vec{z}_{Ph}}$$

with

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \quad \left(\text{a "metric in phase space"}, J^{-1} = -J\right)$$

A canonical transformation $\vec{z} \to \vec{z}'$ preserves the Hamilton equation:

$$\begin{aligned} M_{\alpha\beta} &:= \frac{\partial z_\alpha}{\partial z'_\beta} \quad \alpha, \beta = 1, \ldots 2N \\ M^{-1}_{\alpha\beta} &:= \frac{\partial z'_\alpha}{\partial z_\beta} \end{aligned}$$

$$\dot{z}'_\alpha = \frac{\partial z'_\alpha}{\partial t} = \frac{\partial z'_\alpha}{\partial z'_\beta}\frac{\partial z_\beta}{\partial t} = \underbrace{\frac{\partial z'_\alpha}{\partial z_\beta}}_{M^{-1}_{\alpha\beta}} J_{\beta\gamma}\frac{\partial \mathbf{H}}{\partial z_\gamma} = M^{-1}_{\alpha\beta} J_{\beta\gamma}\frac{\partial \mathbf{H}}{\partial z'_\delta}\underbrace{\frac{\partial z'_\delta}{\partial z_\gamma}}_{M^{-1}_{\delta\gamma}} \stackrel{!}{=} J_{\alpha\delta}\frac{\partial \mathbf{H}}{\partial z'_\delta} \tag{1.27}$$

$$\Rightarrow \quad M^{-1}_{\alpha\beta} J_{\beta\gamma} {M^T}^{-1}_{\gamma\delta} = J_{\alpha\delta}$$

$$\boxed{\Rightarrow \quad J = MJM^T} \tag{1.28}$$

M is a *symplectic matrix* (the M form a group)!

- The Poisson bracket

$$\{f, g\}_{Poisson}(z) = -\frac{\partial f}{\partial z_\alpha} J_{\alpha\beta} \frac{\partial g}{\partial z_\beta}$$

is invariant under canonical (symplectic) transformations.

1.6 The closed oscillator chain

1.6.1 The classical system

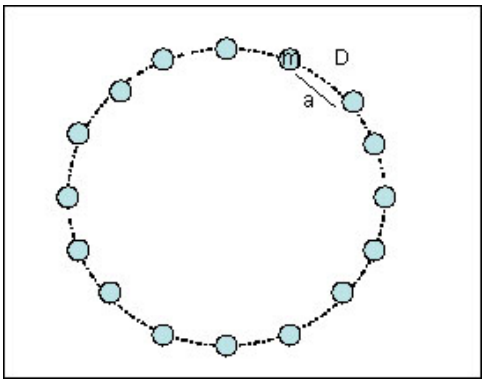


Figure 1.2: The closed oscillator chain

Let us consider a circular chain of radius R, with N masses m, connected by springs with spring constant D. Let the ith mass be at a displacement $q_i = R\Theta_i$ from some rest position, and let us impose the periodic boundary condition $q_0 \equiv q_N$ for convenience of writing. The distance between the masses when they are at rest is $a = 2\pi R/N$.

The Lagrangian for this system is:

$$L(q,\dot{q}) = \sum_{j=1}^{N} \left[\frac{m}{2}\dot{q}_j^2 - \frac{D}{2}\left(q_{j+1} - q_j\right)^2 \right]$$

Varying q_l, we get the equation of motion

$$m\ddot{q}_l + D\left(q_l - q_{l-1} - (q_{l+1} - q_l)\right) = 0$$

which is solved by the ansatz

$$q_l(t) = e^{i(lk-\omega t)} \quad (+vt)$$

leading to

$$m\omega^2 + \underbrace{D}_{m\bar{\omega}^2} \underbrace{\left(2 - e^{-ik} - e^{+ik}\right)}_{4\sin^2\frac{k}{2}} = 0$$

Because of the periodic boundary condition $q_N = q_0$, we get $Nk = 2\pi n$ with $n = 0, \ldots, N-1$ or $n = -N/2, \ldots, +N/2 - 1$ (for even N), and we obtain for the *oscillator modes*

$$\omega_n^2 = \underbrace{\frac{4D}{m}}_{4\bar{\omega}^2} \sin^2 \frac{\pi n}{N} \tag{1.29}$$

Superposition of modes gives the general solution for real $q_j(t)$:

$$q_j(t) = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} \left[c_n e^{i(2\pi nj/N - \omega_n t)} + c_n^* e^{-i(2\pi nj/N - \omega_n t)} \right] \tag{1.30}$$

(note that this is a Fourier expansion).
The Hamiltonian is then given by:

$$\mathbf{H}(q,p) = \sum_{j=1}^{N} \left[\frac{p_j^2}{2m} + \frac{D}{2} (q_j - q_{j-1})^2 \right],$$

where $p_j = m\dot{q}_j$, and the corresponding equations of motion are

$$\dot{p}_j = -\frac{\partial \mathbf{H}}{\partial q_j}, \quad \dot{q}_j = \frac{\partial \mathbf{H}}{\partial p_j}.$$

To ensure that the the transformation $(q_j, p_j) \to (c_n, c_n^*)$ is canonical, we normalize c_n and c_n^* by defining the new canonical variables a_n and ia_n^* through $a_n = \sqrt{2m\omega_n} c_n e^{-i\omega_n t}$ (cf. harmonic oscillator, sec. 1.5.1). These variables are canonical:

$$\{a_n, ia_m^*\}_{Poisson} = -\delta_{nm}, \quad \{a, a\} = \{a^*, a^*\} = 0$$

The new Hamiltonian is

$$H = \sum_n \frac{1}{2} \omega_n (a_n^* a_n + a_n a_n^*)$$

Note that this transformation is still classical. It is preferable, however, to discuss this in the already quantized version.

1.6.2 Quantization

The procedure is exactly the same as it was for the harmonic oscillator:

$$\begin{aligned} p_l, q_l &\to \mathbf{p}_l, \mathbf{q}_l \\ \{f(p,q), g(p,q)\}_{Poisson} &\to \frac{i}{\hbar} [\mathbf{f}, \mathbf{g}] \\ a_n, a_n^* &\to \bar{\mathbf{a}}_n, \bar{\mathbf{a}}_n^\dagger \end{aligned}$$

Then the $\bar{\mathbf{a}}_n$, $i\bar{\mathbf{a}}_n^\dagger$ fulfill the usual commutation relation

$$\left[\bar{\mathbf{a}}_n, \bar{\mathbf{a}}_m^\dagger\right] = i\hbar\delta_{nm}$$

and the Hamiltonian looks like this:

$$\mathbf{H} = \sum_n \omega_n \left(\bar{\mathbf{a}}_n^\dagger \bar{\mathbf{a}}_n + \frac{1}{2}\hbar\right)$$

Exercise: obtain the commutator $[p_j, q_{j'}]$ from $[\bar{\mathbf{a}}_n, \bar{\mathbf{a}}_m^\dagger] = i\hbar\delta_{nm}$ and derive the Hamiltonian above (use $\sum_{n=0}^{N-1} e^{2\pi i n(j-j')/N} = N\delta_{jj'}$).

In the following we will often use units where $\hbar = 1$. We will sometimes reinsert the factors of $\hbar$, in cases with experimental relevance; the same goes for setting $c = 1$.

Defining $\mathbf{a}_n^{(\dagger)} = \sqrt{\hbar}\bar{\mathbf{a}}_n^{(\dagger)}$, the Hamiltonian becomes:

$$\mathbf{H} = \sum_n \hbar\omega_n \left(\mathbf{a}_n^\dagger \mathbf{a}_n + \frac{1}{2}\right) \tag{1.31}$$

After quantization each mode n has an excitation number (the "occupation number" of some quasiparticle-mode), as we have already seen in the case of the harmonic oscillator. This begs the question whether, as a physicist, one can distiguish these modes (called *phonons* if they live in crystals) from "real" particles.

If one can see the "lattice" (e.g. a solid state crystal), the answer is yes, because a real particle should exist independently of the medium it lives in. One can not always see this medium, however, in particular when the continuum limit has been taken. Then, the "lattice" has become some sort of "ether", which is not necessarily visible to the physicist, and might not be needed anymore.

An interesting case is fundamental (super)string theory, where particles are modes of strings.

The states on which the operators defined above act live in a *Fock space* $\mathfrak{F} = \mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \cdots \otimes \mathfrak{H}_N$, where $\mathfrak{H}_i$ the Hilbert space of i-th mode. The "vacuum" state is denoted by $|0\rangle$, and has $a_m |0\rangle = 0$ for all m.

General states

$$|n_1, n_2, \ldots, n_N\rangle = \frac{(\mathbf{a}_1^\dagger)^{n_1}}{\sqrt{n_1}} \frac{(\mathbf{a}_2^\dagger)^{n_2}}{\sqrt{n_2}} \cdots \frac{(\mathbf{a}_N^\dagger)^{n_N}}{\sqrt{n_N}} |0\rangle$$

are normalized eigenstates of $\mathbf{H}$ and have energy

$$E = \sum_m \hbar\omega_m \left(n_m + \frac{1}{2}\right)$$

Note: Elastic binding to lattice, mimicking a crystal, at $x_j = j2\pi R/N$ gives a term $-(m/2)\Omega^2 q_j^2$ in L which shifts the ω_n^2 in eq. (1.29) by $+\Omega^2$; there is no zero mode.

One can rewrite eq. (1.30) by substituting $-n$ for n in the second term:

$$q_j = \sum_{n=-N/2}^{N/2-1} e^{2\pi inj/N} \left[\frac{\mathbf{a}_n + \mathbf{a}_{-n}^\dagger}{\sqrt{2m\omega_n}} \right] = \sum_{n=-N/2}^{N/2-1} e^{2\pi inj/N} \mathbf{Q}_n$$

Similarly, one can introduce

$$\mathbf{P}_n = \sqrt{\frac{m\omega_n}{2}} i(\mathbf{a}_n + \mathbf{a}_{-n}^\dagger)$$

Check the following relations:

$$[\mathbf{P}_{-n}, \mathbf{Q}_l] = -i\delta_{nl}$$

and

$$\mathbf{H} = \sum_n \frac{1}{2m} \mathbf{P}_{-n}\mathbf{P}_n + \frac{1}{2} m\omega_n^2 \mathbf{Q}_{-n}\mathbf{Q}_n$$

with $\mathbf{P}_n = \mathbf{P}_n^\dagger$ and $\mathbf{Q}_n = \mathbf{Q}_n^\dagger$

1.6.3 Continuum limit

In the continuum limit, we take N to infinity and a to zero while keeping $2\pi R = Na$ fixed. The index j becomes a continuous variable, leading to:

$$q_j(t) \to q(x,t), \quad x = a \cdot j \tag{1.32}$$

Note that in the "closed string" case discussed above, one does not have elastic binding to the lattice like one has for a crystal, and x is defined up to an overall translation.

Now, we make the following translations:

$$\begin{aligned} (q_j - q_{j-1})^2 &\to a^2 \left(\frac{\partial q(x,t)}{\partial x} \right) \\ \sum_j &\to \frac{1}{a} \int_0^{2\pi R} \mathrm{d}x \end{aligned}$$

Further, we postulate

$$\begin{aligned} m &= \rho a && \text{with finite mass density } \rho \\ D &= \frac{\sigma}{a} && \text{with finite string density } \sigma \end{aligned}$$

and obtain the following Lagrangian:

$$L^{cont} = \frac{1}{2}\int_0^{2\pi R} \mathrm{d}x \left[\rho\left(\frac{\partial q(x,t)}{\partial t}\right)^2 - \sigma\left(\frac{\partial q(x,t)}{\partial x}\right)^2\right] \tag{1.33}$$

The equations of motion

$$\rho\frac{\partial^2 q(x,t)}{\partial t^2} - \sigma\frac{\partial^2 q(x,t)}{\partial x^2} = 0$$

can be read off from the discontinous case. Later, we will obtain them directly from eq. (1.33).

Now, with $c = \sqrt{\sigma/\rho}$ we can rewrite the above equation as:

$$\frac{\partial^2 q}{\partial t^2} - c^2\frac{\partial^2 q}{\partial x^2} = 0 \tag{1.34}$$

Note: introducing elastic binding (cf. note in section 1.6.2) gives an additional mass term $-\frac{1}{2}\int_0^{2\pi R} dx\rho\Omega^2 q^2(x,t)$ in eq. (1.33), and $-\Omega^2 q$ in eq. (1.34), which then has the form of a Klein-Gordon equation to be discussed in the next chapter.

Now, we can just translate our "discrete" solution to the continuum case, *or* we can make the following ansatz (from electrodynamics):

$$q(x,t) = Ae^{i(kx-\omega t)}$$

Plugging this into eq. (1.34) gives

$$\omega^2 = c^2k^2$$

Imposing periodic boundary conditions $q(0,t) = q(2\pi R,t)$ leads to

$$k_n = \frac{n}{R} \quad (n = 0, \pm1, \pm2, \dots)$$

which, in combination with $\omega^2 = c^2k^2$, gives

$$\omega_n = \frac{|n|c}{R} \tag{1.35}$$

Note that in the finite N case, the continuum ω_n is only approximately valid. We have to use the formula from the discrete case:

$$\omega_n = \sqrt{\frac{4D}{m}}\sin\left(\frac{\pi|n|}{N}\right) \tag{1.36}$$

We expand this for small n, using the linear approximation for the sine:

$$\omega_n \sim \sqrt{\frac{4D}{m}}\frac{\pi|n|}{N} = \frac{2c\pi|n|}{aN} = \frac{c|n|}{R}$$

(remember $D = c^2 m/a^2$ and $a = 2\pi R/N$). This approximation is only allowed when $\frac{\pi|n|}{N} < \frac{\pi}{4}$; then, we have:

$$\omega_n \leq \frac{c}{R}\frac{N}{4} = \omega_c \tag{1.37}$$

Note that in the above discussion, c does not always have to be the speed of light; it may also be a characteristic speed for the medium in which our phonons are living.

1.6.4 Ground state energy

$$E_0 = \sum_n \frac{\hbar\omega_n}{2} = \sum_{n=-\infty}^{+\infty} \hbar\frac{|n|c}{2R} \tag{1.38}$$

In the continuum limit the ground state energy is divergent! We will observe such divergences more often in QFT; they are related to the continuum limit and to interaction at a point, and require physical discussion.

In the finite N case:

$$\begin{aligned}
E_0 &= \frac{\hbar}{2}\sum_{n=-N/2}^{N/2-1}\frac{c}{R}\frac{N}{\pi}\sin\left(\frac{\pi|n|}{N}\right) \\
&\approx \frac{\hbar}{2}\int_{-1/2}^{1/2} \mathrm{d}x \frac{c}{R}\frac{N^2}{\pi}\sin(\pi|x|) \quad (\text{use } x \sim \frac{n}{N}) \\
\frac{E_0}{2\pi R} &\approx \frac{\hbar}{2}\cdot 2\cdot\frac{1}{\pi}\cdot\frac{cN^2}{\pi R\cdot 2\pi R} \\
&= \frac{\hbar c}{2\pi^3}\frac{N^2}{R^2} = \frac{2\hbar c}{\pi a^2} \quad \text{with} \quad \frac{N^2}{R^2} = \left(\frac{2\pi}{a}\right)^2
\end{aligned}$$

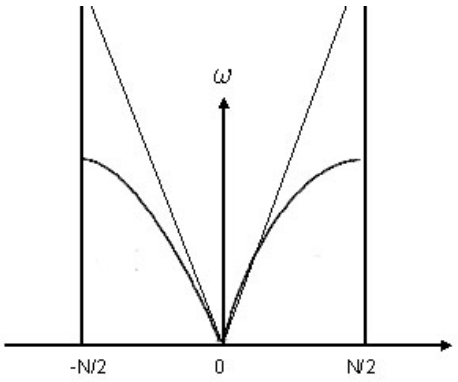
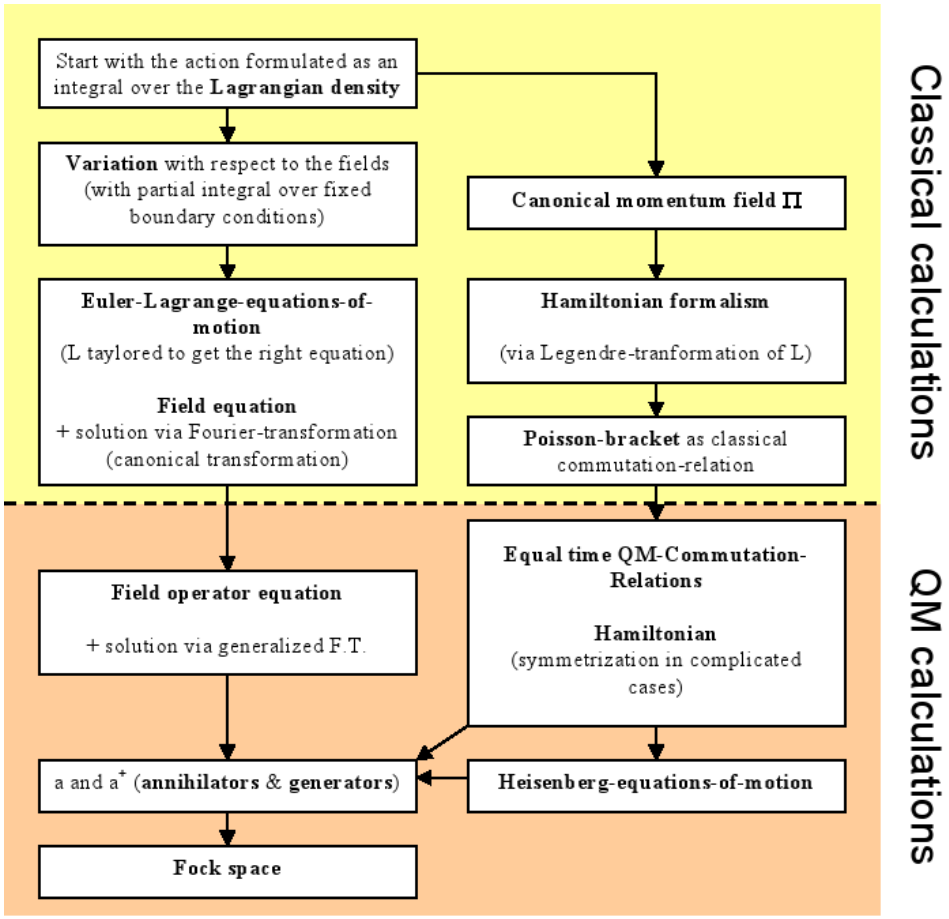


Figure 1.3: Validity range for the linear approximation

The absolute value of E_0 does not have a physical meaning (except in general relativity, where it appears on the right hand side of the Einstein equation), but we can compare ground state energy in two *different* physical situations; in doing so, we can observe the Casimir effect.

1.7 Summary



1.8 Literature

Literature list for QFT I and II; books marked with * are particularly recommended for studies accompanying the lectures

*L. H. Ryder	Quantum Field Theory (Cambridge Univ. Press) rather elementary, clear, wide variety of subjects
*M. E. Peskin, D. E. Schroeder	An introduction to QFT (Addison-Wesley)
*M. Maggiore	A modern Introduction to QFT (Oxford Univ. Press)
*C. Itzykson, J. B. Zuber	Quantum Field Theory (Mc Graw Hill, also paperback) comprehensive standard book, not always up to date, hard to read in a row (gauge theories)
*P. Ramond	Field Theory, a modern Primer (Frontiers of Physics, Benjamin) early book on path integral formulation, non abelian gauge theory, unconventional
*S. Weinberg	The Quantum Theory of Fields I, II very extensive, sometimes non-standard conventions
*K. Huang	Quantum Field Theory: from operators to Path integrals (Wiley Interscience) looking to QFT outside elementary particle physics see also Quarks... (World Scientific Publish.)
*M. Stone	The Physics of Quantum Fields (Springer) Elementary, wide range
T. Kugo	Eichtheorie (Springer) Thorough discussion of gauge theory quantization
E. Harris	A Pedestrian Approach to QFT (Wiley, also in German) rather elementary, short
F. Mandl, G. Shaw	Quantum Field Theory (Wiley Interscience) Elementary introduction
V. P. Nair	Quantum Field Theory, A Modern Perspective (Springer) Interesting advanced chapters
A. Zee	Quantum Field Theory in a Nutshell (Princeton Univ. Press) unconventional presentation of concepts, diverse range of topics
L. S. Brown	Quantum Field Theory (Cambridge University Press)
S. J. Chang	Introduction to QFT (World Scientific, Lecture Notes in Physics, Vol. 29)
J. D. Björken, S. Drell	*I. Rel. Quantenmechanik II. Rel. Quantentheorie (BI 98 and 101) (II. outdated, no QCD) a standard book some decades ago
N. N. Bogoliubov, D. V. Shirkov	Quantum Fields (Benjamini, Cummings;..) related to the famous monography of the same authors, which today is partly outdated

L. D. Fadeev, A. A. Starnov	Gauge Fields, Introduction to QFT (Benjamini, Cummings) advanced studies
R. J. Rivers	Path Int. Methods in QFT (Cambdridge Mon. on Math. Phys.)
J. Zinn-Justin	Quantum Field Theory and Critical Phenomena (Oxford Science Publications) the connection to statistical mechanics
I. J. R. Aitchison	Relativistic Quantum Mechanics (Mac Millan) strong relation to elementary particle physics, elementary
C. Nash	Relativistic Quantum Fields (Academic Press) renormalization theory for ϕ^4
R. P. Feynman	Quantum Electrodynamic (Frontiers in physics, Benjamin. inl.) unconventional, somewhat outdated
J. M. Jauch, F. Rohrlich	The Theory of Photons and Electrons (Springer) old QED-monography
A. L. Fetter, J. D. Walecka	Quantum theory of Many Particle Systems (McGraw Hill) Feynman diagrams in solid-state physics
A. Smilga	Lectures on Quantum Chromodynamics (World Scientific)
M. Le Bellac	Quantum and Statistical Theory (Clarence Press, Oxford) Thermal QFT

Chapter 2

The free electromagnetic field; Klein-Gordon equation; quantization

We quantize the free electromagnetic field in the Coulomb/radiation gauge, analogously to the oscillator string. Then, we discuss electromagnetic emission and absorption in this language. We introduce the Lagrange- and Hamilton-formalisms for fields and present a more thorough discussion of the canonical quantization of the Klein-Gordon equation. To conclude, we come back to the Maxwell equations in the canonical formalism.

2.1 The free electromagnetic field

2.1.1 Classical Maxwell theory

The classical source-free Maxwell equations are:

$$\vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\cdot\vec{E}=0,\quad \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial}{\partial t}\vec{B},\quad \vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial}{\partial t}\vec{E} \tag{2.1}$$

In the Coulomb/radiation gauge, we have

$$\vec{\nabla}\cdot\vec{A}=0,\quad \phi=0,\quad \text{so}\quad \vec{E}=-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

Using $\vec{B}=\vec{\nabla}\times\vec{A}$ and Ampère's law, we obtain

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A}\right)}_{=0}-\vec{\nabla}\cdot\vec{\nabla}\vec{A}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

resulting in

$$\boxed{\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{x},t) = 0} \tag{2.2}$$

for the wave equation.

In order to solve the wave equation in a finite volume L^3 with periodic boundary conditions, we make the following ansatz, based on a Fourier transform (later on called *Fourier ansatz*):

$$\boxed{\vec{A}(\vec{x},t) = \frac{1}{\sqrt{L^3}}\sum_{\vec{k}} N_{\vec{k}}\left(\vec{a}_{\vec{k}}(t)\, e^{i\vec{k}\vec{x}} + \vec{a}^{*}_{\vec{k}}(t)\, e^{-i\vec{k}\vec{x}}\right)} \tag{2.3}$$

with normalization $N_{\vec{k}}$ and discrete $k_i L = 2\pi n_i$.

From $\vec{\nabla}\cdot\vec{A} = 0$, we have $\vec{k}\cdot\vec{a}_{\vec{k}} = 0$, i.e. the solution has only transversal modes; this is the origin of the name *transversal gauge*. The wave equation gives $\vec{a}_{\vec{k}} \propto e^{i\omega t}$, with dispersion relation $c|\vec{k}| = \omega$. Now, let us calculate the energy of the electromagnetic field:

$$\begin{aligned} E &= \frac{1}{8\pi}\int \mathrm{d}^3x\, \left(\vec{E}^2 + \vec{B}^2\right) \\ &= \frac{1}{8\pi}\int \mathrm{d}^3x \left(\frac{1}{c^2}\left(\frac{\partial}{\partial t}\vec{A}\right)^2 + \left(\vec{\nabla}\times\vec{A}\right)\cdot\left(\vec{\nabla}\times\vec{A}\right)\right) \end{aligned}$$

Through partial integration, the second term becomes

$$\begin{aligned} (-1)^2\left(\vec{A}\times\vec{\nabla}\right)\cdot\left(\vec{\nabla}\times\vec{A}\right) &= \vec{A}\cdot\left(\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)\right) \\ &\downarrow \quad \text{Coulomb gauge} \\ &= -\vec{A}\cdot\Delta\vec{A} \\ &\downarrow \quad \text{partial integration} \\ &= \underbrace{(\vec{\nabla}\vec{A})\cdot(\vec{\nabla}\vec{A})}_{\partial_i A_k \partial_i A_k} \end{aligned}$$

leading to

$$\boxed{E = \frac{1}{8\pi}\int \mathrm{d}^3x \sum_{i=1}^{3}\left(\frac{1}{c^2}\left(\frac{\partial}{\partial t}A_i\right)^2 + \left(\vec{\nabla}A_i\right)^2\right)} \tag{2.4}$$

Substituting the Fourier ansatz gives

$$E = \frac{2L^3}{8\pi L^3}\frac{1}{c^2}\sum_{\vec{k}} \omega^2_{\vec{k}}\, N^2(\omega_{\vec{k}})\left(\vec{a}_{\vec{k}}\,\vec{a}^{*}_{\vec{k}} + \vec{a}^{*}_{\vec{k}}\,\vec{a}_{\vec{k}}\right)$$

Exercise: provide the intermediate steps. Now, using $\omega = c|\vec{k}|$ and orthogonality, we have

$$\int_V \mathrm{d}^3x \; e^{i(\vec{k}-\vec{k}')\vec{x}} = L^3 \delta_{\vec{k},\vec{k}'}$$

where a *Kronecker* delta appears because the k-indices are discrete (remember that we had periodic boundary conditions).

The cross terms $(\vec{a}_{\vec{k}}\, \vec{a}_{-\vec{k}} + \vec{a}^*_{\vec{k}}\, \vec{a}^*_{-\vec{k}})$ in the sum come from both $(\partial\vec{A}/\partial t)^2/c^2$ and $(\vec{\nabla}\vec{A})$, but with opposite sign, so they cancel. Finally, choosing $N = \sqrt{2\pi c^2/\omega_k}$, we obtain

$$\boxed{E = \frac{1}{2}\sum_{\vec{k}} \omega_{\vec{k}} \left(\vec{a}_{\vec{k}}\, \vec{a}^*_{\vec{k}} + \vec{a}^*_{\vec{k}}\, \vec{a}_{\vec{k}}\right) \quad \text{with } \vec{k}\cdot\vec{a}_{\vec{k}} = 0} \tag{2.5}$$

which is a sum over harmonic oscillator terms.

The Poynting vector is calculated similarly:

$$\vec{P} = \frac{1}{4\pi c}\int \mathrm{d}^3x \; \vec{E}\times\vec{B} = \sum_{\vec{k}} \vec{k}\vec{a}^*_{\vec{k}} \cdot \vec{a}_{\vec{k}}$$

which, with $\vec{E}\times\vec{B} = -\frac{1}{c}\left(\frac{\partial}{\partial t}\vec{A}\right)\times\left(\vec{\nabla}\times\vec{A}\right)$, becomes:

$$\vec{P} = -\frac{1}{c}\left(\vec{\nabla}\left(\frac{\partial}{\partial t}\vec{A}\right)\cdot\vec{A} - \left(\frac{\partial}{\partial t}\vec{A}\right)\cdot\vec{\nabla}\vec{A}\right)$$

The second term becomes zero after partial integration.

Note

- In the continuum limit, where $L\to\infty$, the sum is replaced by an integral: $\Sigma \to \int \mathrm{d}^3k/(2\pi)^3 L^3$. With $N \propto \frac{1}{\sqrt{L^3}}$, it follows $A \propto \int \mathrm{d}^3k/(2\pi)^3(...)$, and $H = 1/2\int \mathrm{d}^3k/(2\pi)^3(...)$

  The expression (2.5) above can be interpreted intuitively as a collection of an infinite number of harmonic oscillators, with $\vec{a}_{\vec{k}}$ and $i\vec{a}^*_{\vec{k}}$ as generalized coordinates and momenta (compare to the oscillator chain from chapter one). We will have to substantiate this heuristic interpretation when introducing the *canonical formalism* for fields in the next section.

- The corresponding position (and momentum) operators cannot naïvely be identified with the argument $\vec{x}$ of $\vec{A}(\vec{x},t)$.

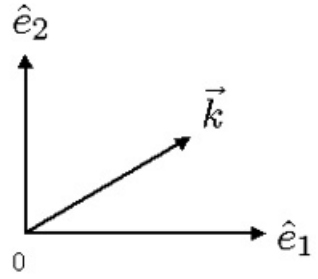


Figure 2.1: Electromagnetic wave propagation

2.1.2 Quantization

The quantization is procedure is similar to the one for harmonic oscillator

$$\boxed{\frac{i}{\hbar}\left[\mathbf{a}^{\sigma}_{\vec{k}}, i\mathbf{a}^{\dagger\lambda}_{\vec{k}'}\right] = -\delta_{\vec{k},\vec{k}'}\delta^{\sigma,\lambda}, \quad \sigma,\lambda = 1,2} \tag{2.6}$$

for *transversal* polarization degrees of freedom σ: $\vec{a}_{\vec{k}} = \vec{e}^{\,1}\, a^1_{\vec{k}} + \vec{e}^{\,2}\, a^2_{\vec{k}}$ with $\vec{e}^{\,1} \perp \vec{e}^{\,2} \perp \vec{k}$.

Promoting the $\vec{a}_{\vec{k}}$ and $i\vec{a}^*_{\vec{k}}$ from eq. (2.5) to operators, we have

$$\mathbf{H} = \frac{1}{2}\sum_{\vec{k},\sigma} \omega_{\vec{k}} \left(\mathbf{a}^{\sigma}_{\vec{k}}\, \mathbf{a}^{\dagger\sigma}_{\vec{k}} + \mathbf{a}^{\dagger\sigma}_{\vec{k}}\, \mathbf{a}^{\sigma}_{\vec{k}}\right)$$

leading to

$$\boxed{\mathbf{H} = \sum_{\vec{k},\sigma} \omega_{\vec{k}} \left(\mathbf{a}^{\dagger\sigma}_{\vec{k}}\, \mathbf{a}^{\sigma}_{\vec{k}} + \frac{1}{2}\hbar\right)} \tag{2.7}$$

Note that $\mathbf{a}^{\dagger\sigma}_{\vec{k}}\, \mathbf{a}^{\sigma}_{\vec{k}} = \hbar \mathbf{n}^{\sigma}_{\vec{k}}$ is the occupation number of the harmonic oscillator $n^{\sigma}_{\vec{k}}$ and the zero-point energy of the $\vec{k}$-mode is $\hbar\,\omega_{\vec{k}}/2$.

Again, the space of states is the direct product space of the Hilbert-spaces corresponding to the modes, called *Fock-space*:

$$\mathfrak{H} = \prod_{\vec{k},\sigma} \otimes\, \mathfrak{H}_{\vec{k},\sigma}\,; \quad \text{states: } \left| n^{\sigma_1}_{\vec{k}}, n^{\sigma_2}_{\vec{k}}, \ldots n^{\sigma_N}_{\vec{k}} \right\rangle$$

$$E = \sum_{\vec{k},\sigma} \hbar\omega_{\vec{k}} \left(n^{\sigma}_{\vec{k}} + \frac{1}{2}\right); \quad \vec{P} = \sum_{\vec{k},\sigma} \hbar\,\vec{k}\, n^{\sigma}_{\vec{k}}$$

Here we have an infinite number of degrees of freedom (modes), each with its own occupation number as for the harmonic oscillator. Because

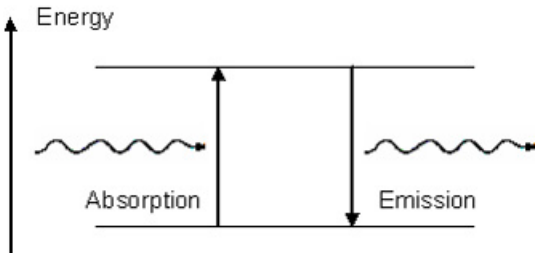


Figure 2.2: Absorption and Emission process

of the sum there is an infinite ground state energy, the so-called vacuum energy. This can be removed it by *normal ordering*, $:\mathbf{a}\mathbf{a}^\dagger: = \mathbf{a}^\dagger\mathbf{a}$ (moving all annihilation operators to the right of all creation operators), but this is ad hoc. Infinite energy shifts are also a problem in general relativity, where energy density influences the curvature of spacetime.

2.1.3 Application: emission, absorption

Consider induced emission or absorption, or spontaneous emission due to an interaction (in QM):

$$\mathbf{H}_I = -\frac{e}{mc}\vec{p}\cdot\vec{A}(\vec{x},t). \tag{2.8}$$

With a quantized electromagnetic field, we have

$$\begin{aligned} \text{initial state} \quad |i\rangle &= |a\rangle_{Atom}\left|\ldots n_{\vec{k}}^{\sigma}\right\rangle \\ \text{final state} \quad |f\rangle &= |b\rangle_{Atom}\left|\ldots n_{\vec{k}}^{\sigma}\pm 1\right\rangle \end{aligned}$$

With

$$\vec{A}(\dot{x},t) = \frac{1}{\sqrt{L^3}}\sum_{\vec{k},\sigma} N\left(a_{\vec{k}}^{\sigma}\,\vec{e}_{\vec{k}}^{\sigma}\,e^{i\vec{k}\vec{x}} + a_{\vec{k}}^{\dagger\sigma}\,\vec{e}_{\vec{k}}^{\sigma}\,e^{-i\vec{k}\vec{x}}\right)$$

we obtain

$$\langle f|\mathbf{H}_I|i\rangle = -\frac{e}{mc}\left(\frac{2\pi\hbar c^2}{L^3\omega_k}\right)^{1/2}\left\langle b|\vec{\mathbf{p}}\cdot\vec{e}_{\vec{k}}^{\sigma}\,e^{\pm i\vec{k}\vec{x}}|a\right\rangle\begin{cases}\sqrt{n_{\vec{k}}^{\sigma}+1}\\ \sqrt{n_{\vec{k}}^{\sigma}}\end{cases}$$

$n_{\vec{k}}^{\sigma} = 0$ means spontaneous emission ($\rightarrow$ "natural line width", considering perturbation theory).

Finally, we have

$$\text{transition probability / time} = \frac{2\pi}{\hbar}\left|\langle f|\mathbf{H}_I|i\rangle\right|^2\delta(E_f - E_i).$$

2.1.4 Coherent states

To approximate an electromagnetic wave one introduces coherent states like for the single harmonic oscillator, and expresses $\vec{A}$, $\vec{E}$, and $\vec{B}$ through expectation values in these coherent states. First observe that with

$$\tilde{\mathbf{E}} = -\frac{1}{c}\frac{\partial}{\partial t}\tilde{\mathbf{A}} = -i\sum_{\vec{k},\sigma}\left(\frac{2\pi\hbar\omega_k}{L^3}\right)^{1/2}\vec{\epsilon}_{\vec{k}}^{\,\sigma}\left(\mathbf{a}_{\vec{k}}^{\sigma}(t)\,e^{i\vec{k}\vec{x}} - \mathbf{a}_{\vec{k}}^{\dagger\sigma}(t)\,e^{-i\vec{k}\vec{x}}\right)$$

$\langle n|\,\tilde{\mathbf{E}}\,|n\rangle = 0$ for $\tilde{\mathbf{E}}$ between states with fixed photon number n. Also note

$$\boxed{\left[\mathbf{N}_{\vec{k}}^{\sigma}, \tilde{\mathbf{E}}\right] \neq 0} \tag{2.9}$$

i.e. photon number and electric field can not be measured simultaneously. However, we also have

$$\left\langle n|\frac{\tilde{\mathbf{E}}^2}{4\pi}|n\right\rangle = \frac{\hbar\omega}{L^3}\left(n+\frac{1}{2}\right) \neq 0.$$

Now consider a monochromatic coherent state

$$\left|\lambda, \vec{k}, \sigma\right\rangle = \mathbf{N}e^{\lambda_{\vec{k},\sigma}(t)\mathbf{a}_{\vec{k}}^{\sigma\,\dagger}}\,|0\rangle_{\vec{k},\sigma}\,. \tag{2.10}$$

and calculate

$$\left\langle \lambda, \vec{k}, \sigma|\tilde{\mathbf{E}}|\lambda, \vec{k}, \sigma\right\rangle = -i\left(\frac{2\pi\hbar\omega}{L^3}\right)^{1/2}\vec{e}_{\vec{k}}^{\,\sigma}\left(\lambda_{\vec{k},\sigma}(t)\,e^{i\vec{k}\vec{x}} - \lambda_{\vec{k},\sigma}^{*}(t)\,e^{-i\vec{k}\vec{x}}\right)$$

which is the Fourier-component of a monochromatic, polarized $\tilde{\mathbf{E}}$. Observe that λ determines the amplitude. The conclusion is that for induced absorption/emission, one can get back the normal QM description by using coherent states. (Note: the "Fock-space vacuum" $|0\rangle = \prod_{\vec{k},\sigma}\otimes|0\rangle_{\vec{k},\sigma}$ allows for product states of (2.10) in the more general case.)

2.2 Klein-Gordon equation

2.2.1 Lagrange formalism for field equations

Consider an action

$$S[\mathcal{L}] = \int_{t_0}^{t_1}\mathrm{d}t\,\underbrace{\int\mathrm{d}^3x\;\mathcal{L}\left(\Phi(\vec{x},t),\left(\frac{\partial}{\partial t}\Phi\right),\vec{\nabla}\Phi\right)}_{L(\Phi(\cdot,t),\dots)}$$

and a *field* $\Phi(\vec{x},t)$. Discretizing the field, using a lattice, turns it into a vector, with position serving as an index: $\Phi(\vec{x}_i,t)\to\Phi_i(t)$. Note, however,

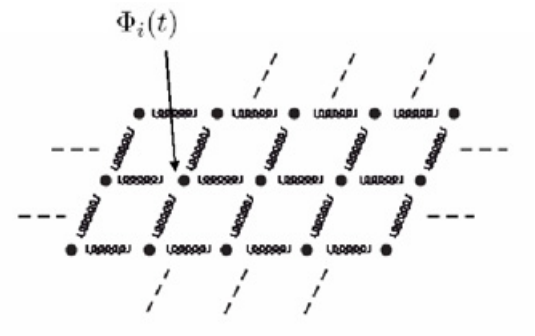


Figure 2.3: Field Lattice

that we retain infinitely many degrees of freedom even in the discretized case.

The Lagrangian L is a functional of $\Phi(\cdot, t)$, the action S of $\Phi(\cdot, \cdot)$. We postulate $\delta S[\mathcal{L}] = 0$ with a *variational derivative*, i.e. S is extremal (changing Φ at position $\vec{x}$ and time t).

Functional derivatives

For a functional $F(\Phi(\cdot))$, we have

$$F(\Phi(\cdot) + \epsilon\xi(\cdot)) - F(\Phi(\cdot)) = \epsilon \int \mathrm{d}x \, \frac{\delta F}{\delta \Phi(x)} \, \xi(x) + O(\epsilon^2)$$

which defines the functional derivative $\frac{\delta F}{\delta \Phi(x)}$. An elegant, if a little heuristic, choice for the perturbation function ξ is $\xi(x) = \delta(x - \bar{x})$.

$$\lim_{\epsilon \to 0} \frac{F(\Phi(x) + \epsilon\delta(x - \bar{x})) - F(\Phi(x))}{\epsilon} = \left. \frac{\delta F}{\delta \Phi(x)} \right|_{x=\bar{x}}$$

Remarks:

- Discretization ($x \to x_i, \Phi(x_i) \to \Phi_i$) turns the derivative into a gradient and $\delta(x - \bar{x})$ into δ_{ij}.
- If the functional is defined as $F(\Phi(\cdot)) = \int \mathrm{d}x \, \mathcal{F}(\Phi(x))$, then

  $$\frac{\delta F(\Phi(\cdot))}{\delta \Phi(x)} = \frac{\partial \mathcal{F}(\Phi(x))}{\partial \Phi(x)}$$

  i.e. we also can get along without functional derivative notation.
- $\Phi(x_a)$ is a functional itself:

  $$\Phi(\cdot) \to \Phi(x_a)$$

$$\left.\frac{\delta\Phi(x_a)}{\delta\Phi(x)}\right|_{x=\bar{x}} = \lim_{\epsilon\to 0}\frac{\Phi(x_a)+\epsilon\delta(x_a-\bar{x})-\Phi(x_a)}{\epsilon} = \delta(x_a-\bar{x})$$

This will be important later on when calculating the Poisson bracket of Φ, Π.

Applying the above to the action, we get:

$$\begin{aligned}\delta S = \int_{t_0}^{t_1} \mathrm{d}t \int \mathrm{d}^3x' \left\{ \frac{\partial\mathcal{L}}{\partial\Phi}\epsilon(t)\delta(\vec{x}'-\vec{x}) + \frac{\partial\mathcal{L}}{\partial\left(\partial\Phi/\partial t\right)}\frac{\partial\epsilon(t)}{\partial t}\delta(\vec{x}'-\vec{x}) \right. \\ \left. + \frac{\partial\mathcal{L}}{\partial\Phi_{|i}}\epsilon(t)\frac{\partial}{\partial x_i'}\delta(\vec{x}'-\vec{x}) \right\} \\ \overset{=}{\text{part.int.}} \int_{t_0}^{t_1} \mathrm{d}t\ \epsilon(t) \int \mathrm{d}^3x'\delta(\vec{x}'-\vec{x}) \left\{ \frac{\partial\mathcal{L}}{\partial\Phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\left(\partial\Phi/\partial t\right)} - \frac{\partial}{\partial x_i'}\frac{\partial\mathcal{L}}{\partial\Phi_{|i}} \right\}\end{aligned}$$

where $\Phi_{|i} := \frac{\partial\Phi}{\partial x_i}$. There are no boundary terms in this formulation with δ-functions and $\epsilon(t_0) = \epsilon(t_1) = 0$. In the original, and more "orthodox", formulation, without delta's, $\Phi \to \Phi + \epsilon\xi(\vec{x},t)$, and for all ξ, $\xi = 0$ at the endpoints t_0 and t_1 and $|\vec{x}| \to \infty$.

Regardless of the formulation, one has

$$\delta S = 0 \quad \forall\, \epsilon(t), \vec{x}.$$

resulting in

$$\boxed{\frac{\partial\mathcal{L}}{\partial\Phi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\partial\Phi/\partial t\right)} - \frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial\Phi_{|i}} = 0} \tag{2.11}$$

for the field equation.

Remark: in case of a finite volume, one takes periodic boundary conditions in x-space.

The momentum field is defined as

$$\Pi(\vec{x},t) = \frac{\partial L}{\partial(\partial\Phi/\partial t)} = \frac{\partial\mathcal{L}}{\partial(\partial\Phi/\partial t)}$$

Note: the (canonical) momentum *field* $\Pi(\vec{x},t)$ is *not* the *momentum* $\mathbf{P}$ *of* the field.

From the Lagrangian density $\mathcal{L}$ and associated momentum field Π we obtain the Hamiltonian by means of a Legendre transform.

$$\mathbf{H} = \int \mathrm{d}^3x \underbrace{\left\{ \Pi(\vec{x},t)\ \frac{\partial}{\partial t}\Phi(\vec{x},t) - \mathcal{L}(\Phi, \frac{\partial}{\partial t}\Phi, \vec{\nabla}\Phi) \right\}}_{\text{Hamiltonian density } \mathcal{H}}$$

The Hamilton equations read

$$\boxed{\frac{\partial\Phi}{\partial t}=\frac{\delta\mathbf{H}}{\delta\Pi}=\frac{\partial\mathcal{H}}{\partial\Pi}\qquad\frac{\partial\Pi}{\partial t}=-\frac{\delta\mathbf{H}}{\delta\Phi}=-\left(\frac{\partial\mathcal{H}}{\partial\Phi}-\partial_{|i}\frac{\partial\mathcal{H}}{\partial\Phi_{|i}}\right)} \tag{2.12}$$

The Poisson bracket is defined analogously to the discrete case:

$$\{A(\Pi,\Phi),B(\Pi,\Phi)\}_{Poisson}:=\int\left(\frac{\delta A}{\delta\Pi(\vec{x})}\frac{\delta B}{\delta\Phi(\vec{x})}-\frac{\delta B}{\delta\Pi(\vec{x})}\frac{\delta A}{\delta\Phi(\vec{x})}\right)\,\mathrm{d}^3x=\delta(\vec{x}-\vec{x}')$$

Exercise: show that this indeed gives a delta function. Note that $\Phi,\Pi(\vec{x})$ are particular functionals, see remark page before.

2.2.2 Wave equation, Klein-Gordon equation

The wave equation

$$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)}_{=\Box}A(\vec{x},t)=0$$

is solved by the Fourier ansatz

$$A=e^{\pm i(\omega t-\vec{k}\vec{x})}\quad\text{with}\quad-\frac{\omega^2}{c^2}+k^2=0\quad.$$

With $E=\hbar\omega$ and $p=\hbar k$, this translates into the photon energy relation:

$$-\frac{E^2}{c^2}+p^2=0$$

Similarly, for massive particles we have

$$-\frac{E^2}{c^2}+p^2+m^2c^2=0$$

or, dividing by $\hbar$

$$-\frac{\omega^2(\vec{k})}{c^2}+k^2+\frac{m^2c^2}{\hbar^2}=0$$

This translates to the differential equation

$$\boxed{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta+\frac{m^2c^2}{\hbar}\right)\Phi(\vec{x},t)=0} \tag{2.13}$$

which is the promised *Klein-Gordon equation* for a scalar field. The equation follows from the Lagrangian

$$\mathcal{L}=\frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2-\left(\vec{\nabla}\Phi\right)^2-\frac{m^2c^2}{\hbar^2}\Phi^2\right].$$

The corresponding momentum field is

$$\Pi(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial}{\partial t}\Phi\right)} = \left(\frac{\partial}{\partial t}\Phi\right)$$

resulting in the following Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}\Phi\right)^2 + \left(\vec{\nabla}\Phi\right)^2 + \frac{m^2c^2}{\hbar^2}\Phi^2\right]$$

which is positive definite. It can be simplified by using the Fourier decomposition of $\Phi(\vec{x},t)$ in a volume $V = L^3$

$$\Phi(\vec{x},t) = \frac{1}{\sqrt{L^3}}\sum_{\vec{k}} N_{\vec{k}} \underbrace{\left(a_{\vec{k}}(t)e^{i\vec{k}\vec{x}} + a^*_{\vec{k}}(t)e^{-i\vec{k}\vec{x}}\right)}_{\Phi_{\vec{k}}(t) \text{ real!}}.$$

It solves the Klein-Gordon equation if

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{k}^2 + \frac{m^2c^2}{\hbar^2}\right)\Phi_{\vec{k}}(t) = 0$$

i.e. if $a_{\vec{k}}(t) = e^{-i\omega(\vec{k})t}a_{\vec{k}}(0)$, where $\vec{k}^2 + (mc/\hbar)^2\omega^2(\vec{k})/c^2 = \vec{k}^2 + \frac{m^2c^2}{\hbar^2}$ Since the differentiation is only with respect to time, factors of $e^{\pm i\vec{k}\vec{x}}$ do not matter in this equation. Plugging in this definition of Φ in $\mathcal{H}$ gives

$$\boxed{\mathbf{H} = \frac{1}{2}\sum_{\vec{k}}\omega(\vec{k})\left(a^*_{\vec{k}}a_{\vec{k}} + a_{\vec{k}}a^*_{\vec{k}}\right)} \tag{2.14}$$

where we have set $N_{\vec{k}} = \left(\frac{c^2}{2\omega_{\vec{k}}}\right)^{1/2}$. As we would expect, $a_{\vec{k}}$ and $ia^*(\vec{k})$ are canonically conjugate variables:

$$\frac{\partial a_{\vec{k}}}{\partial t} = \frac{\partial \mathbf{H}}{\partial(ia^*_k)} = -i\omega(\vec{k})\,a_{\vec{k}} \qquad (ia^*_{\vec{k}}) = -\frac{\partial \mathbf{H}}{\partial a_{\vec{k}}} = -\omega_{\vec{k}}\,a^*_{\vec{k}}$$

which can be checked using the form of $a_{\vec{k}}$ derived above.

2.2.3 Quantization

This is done most easily in momentum space. The commutator of the canonically conjugate variables is

$$\frac{i}{\hbar}\left[i\mathbf{a}^\dagger_{\vec{k}'}, \mathbf{a}_{\vec{k}}\right] = \delta_{\vec{k},\vec{k}'} \quad \text{leading to} \quad \left[\mathbf{a}_{\vec{k}'}, \mathbf{a}^\dagger_{\vec{k}}\right] = \hbar\delta_{\vec{k},\vec{k}'} \tag{2.15}$$

We redefine this

$$\mathbf{a}_{\vec{k}} \to \frac{\mathbf{a}_{\vec{k}}}{\sqrt{L^3}}: \quad \left[\mathbf{a}_{\vec{k}'}, \mathbf{a}^\dagger_{\vec{k}}\right] = \hbar L^3\delta_{\vec{k},\vec{k}'}$$

(again, see the oscillator chain). Since we have for $L \to \infty$

$$\sum_{\vec{k}\sim\vec{n}} \delta_{\vec{k},\vec{k}'} = 1 \quad \Rightarrow \quad \int \frac{\mathrm{d}^3k}{(2\pi)^3} L^3 \frac{\delta_3(\vec{k}-\vec{k}')(2\pi)^3}{L^3} = 1$$

we see that in the continuous case, the commutator must be

$$\boxed{[\mathbf{a}_{\vec{k}'}, \mathbf{a}^\dagger_{\vec{k}}] = \hbar(2\pi)^3 \delta_3(\vec{k}-\vec{k}')} \tag{2.16}$$

We could have derived this directly from the canonical formalism, promoting the Poisson bracket to a commutator:

$$\begin{aligned} \left\{\Phi(\vec{x},t), \Pi(\vec{x}',t)\right\}_{Poisson} &= -\delta^3(\vec{x}-\vec{x}') \\ \hbar/i \quad &\Downarrow \qquad \text{quantization} \\ \left[\Phi(\vec{x},t), \Pi(\vec{x}',t)\right] &= i\hbar\delta_3(\vec{x}-\vec{x}') \end{aligned}$$

from which the above result for $\mathbf{a}_{\vec{k}}$ follows. Exercise: invert the relation between Φ, Π and $\mathbf{a}, i\mathbf{a}^\dagger$ and calculate $\left[\mathbf{a}, \mathbf{a}^\dagger\right]$. Alternatively, which is easier, plug in Φ, Π in terms of $\mathbf{a}$, $\mathbf{a}^\dagger$ and obtain the relation above.

Of course we have now

$$\Phi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{k})}} \left(\mathbf{a}_{\vec{k}}(t)\, e^{i\vec{k}\vec{x}} + \mathbf{a}^\dagger_{\vec{k}}(t)\, e^{-i\vec{k}\vec{x}}\right) \tag{2.17}$$

$$\mathbf{H} = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2}\omega(\vec{k}) \left(\mathbf{a}^\dagger_{\vec{k}}(t)\mathbf{a}_{\vec{k}}(t) + \mathbf{a}_{\vec{k}}(t)\mathbf{a}^\dagger_{\vec{k}}(t)\right) \tag{2.18}$$

Remarks

Since we know the wave equation from electromagnetics, and since we used a relativistic dispersion relation between ω and $\vec{k}$ to begin with, it is clear that our equation is relativistically invariant, as well as the Lagrangian related to it. We will learn more about this later on. The remarkable aspect of this whole episode is that canonical quantization, where time plays a special role, can be brought into consistency with relativity, where time and space are on the same footing.

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\Phi\,\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 \ldots$$

2.2.4 The Maxwell equations in Lagrange formalism

Here, we will shortly discuss gauge theories (again, more details will follow). For now, we will restrict ourselves to the free case, i.e. without sources. We will use units where $x^0 = t, c = 1$. Consider the electromagnetic tensor:

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu \tag{2.19}$$

and note that it contains $\vec{E}$ and $\vec{B}$ in *relativistic* notation.

The electric and magnetic fields appear in it as follows:

$$\begin{aligned} F^{0i} &= E^i; \quad F^{ik} = \epsilon^{ikl} B^l \\ A^0 &= \phi; \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}; \quad \vec{B} = \vec{\nabla} \times \vec{A} \end{aligned}$$

with $\vec{\nabla} = \frac{\partial}{\partial x^i} = \partial_i = -\partial^i$. So, two Maxwell equations are automatically fulfilled. What remains to be shown is

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$$

or, in other symbols,

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = 0$$

With eq. (2.5), this becomes

$$\boxed{\partial^2 A^\mu - \partial_\nu \partial^\mu A^\nu = 0} \tag{2.20}$$

In the Coulomb gauge, $\partial_i A^i = 0$ and $\phi = A^0 = 0$ (hence the term *radiation* gauge). Hence, eq. (2.20) reduces to $\partial^2 A^\mu = 0$, which is just the wave equation, as $\partial^2 = \partial/\partial x^{0\,2} - \vec{\nabla}^2$. The equation for A^μ follows from $\mathcal{L} = -1/4 F_{\mu\nu} F^{\mu\nu}$ if one varies the A^μ and A^ν *independently* (Exercise: work this out.)

Note: in terms of $\vec{E}$ and $\vec{B}$, $\mathcal{L} = 1/2(\vec{E}^2 - \vec{B}^2)$, but we do not vary with respect to these variables.

Calculating the canonical momenta (see Björken and Drell, vol. II) gives:

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial A_0/\partial t)} = 0; \quad \Pi^k = \frac{\partial \mathcal{L}}{\partial(\partial A_k/\partial t)} = -\frac{\partial A^k}{\partial t} - \frac{\partial A_0}{\partial x^k} = E^k$$

Inspecting the canonical formalism, we get the following result, which looks reasonable:

$$\begin{aligned} \mathcal{H} &= \Pi^k \frac{\partial}{\partial t} A_k - \mathcal{L} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \vec{E} \cdot \vec{\nabla}\phi \\ \mathbf{H} &= \int \mathrm{d}^3x \; \mathcal{H} = \frac{1}{2} \int \mathrm{d}^3x \; (\vec{E}^2 + \vec{B}^2) \end{aligned}$$

The last term can be disposed of through partial integration, since $\vec{\nabla}\vec{E} = 0$. This is not without problems, however, since $\vec{\nabla}\vec{E} = 0$ is violated by both the fact that $\Pi^0 = 0$, and the canonical commutator of A^k and Π^k, which is given by

$$\left[\Pi^i(\vec{x}, t), A^k(\vec{x}', t)\right] \stackrel{?}{=} -i\delta(\vec{x} - \vec{x}') \; \delta^{ik}$$

Indeed it turns out that in this case, we have to quantize with *constraints* in a more rigorous approach (see the literature on the Dirac bracket, which is

a special line of research). Note that working with constraints already pops up in classical field equations.

The simplest implementation of this idea is to do like is done in QED: take the Coulomb gauge.

$$\begin{aligned} L &= \int \mathrm{d}^3x\, \mathcal{L} = \int \left(-\frac{1}{2} \frac{\partial A_\mu}{\partial x^\nu} \frac{\partial A^\mu}{\partial x_\nu} + \frac{1}{2} \frac{\partial A_\nu}{\partial x^\mu} \frac{\partial A^\mu}{\partial x_\nu} \right) \mathrm{d}^3x \\ &\downarrow \quad \text{Last term: partial integration, use } \partial_i A_i = 0, A_0 = 0 \rightarrow = 0 \\ &= \frac{1}{2} \int \partial_\nu \vec{A}\, \partial^\nu \vec{A}\, \mathrm{d}^3x. \end{aligned}$$

We get 3 times the scalar case. Varying this separately for the two *transversal* and one longitudinal ($\vec{A}_{\text{long}}$) components of $\vec{A}$ with the constraint that $\vec{\nabla} \cdot \vec{A} = 0$, leads us to the canonical picture for $\vec{E}_{\text{trans}}(\vec{\nabla} \cdot \vec{E} = 0)$ and for $\vec{A}_{\text{trans}}(\vec{\nabla} \cdot \vec{A} = 0)$, as was used in heuristic exposé.

Chapter 3

The Schrödinger equation in the language of QFT

3.1 Second Quantization

For radiative transitions, the electromagnetic field is described by an operator in Fock space, whereas the atom retains its 'old' QM description, i.e. with an $e^{i\vec{k}\vec{\mathbf{X}}}$, where the position is the operator $\vec{\mathbf{X}}$.

Now, the following question comes up: can we treat the Schrödinger equation, written down according to the rules of the correspondence principle, in its x-space form, like an ordinary field equation and *quantize* it? This sounds odd, since it is already quantized; why do it again? The point is that we want QM to be a special case of QFT, as mentioned in chapter 1, and therefore, it should be possible to express it in the language of QFT. This is called *second quantization*.

Note that we are not looking for new physics: QM has been tested thoroughly, and looks far too nice to throw away.

3.1.1 Schrödinger equation

In classical mechanics, we have $E = \vec{p}^2/2m + V(\vec{x})$; following the correspondence principle, this gives

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{(x)},t) = \underbrace{\left[\left(\frac{-\hbar^2\vec{\nabla}^2}{2m}\right) + V(\vec{x})\right]}_{H_{Schr}}\psi(\vec{x},t) \tag{3.1}$$

This has stationary solutions $\psi_n = \phi_n(\vec{x})e^{-iE_nt/\hbar}$; we will also use $E_n/\hbar = \omega_n$. General solutions, obtained from a generalized Fourier transform, are

$$\psi(\vec{x},t) = \sum_n a_n(t)\phi_n(\vec{x}) \tag{3.2}$$

where $a_n(t) \propto e^{-i\omega_n t}$ and the $\{\phi_n\}$ form a complete set of orthonormal functions.

In the QFT quantization, the functions $a_n(t)$ are promoted to operators $\mathbf{a}_n(t)$; as usual, we are in the Heisenberg picture. The final result of this second quantization will be:

$$\boxed{\mathbf{H} = \int d^3x \psi^\dagger_{op}(\vec{x},t)\mathcal{H}_{Schr}\psi_{op}(\vec{x},t)} \tag{3.3}$$

which is an operator in Fock space. In terms of $\mathbf{a}_n$, we have

$$\mathbf{H} = \sum_n E_n \mathbf{a}^\dagger_n \mathbf{a}_n \tag{3.4}$$

where the $\mathbf{a}_n$ and $i\mathbf{a}^\dagger_n$ are canonically conjugate operators in the Hamiltonian formalism. Let us now consider in detail how we get this result.

3.1.2 Lagrange formalism for the Schrödinger equation

In order to quantize the Schrödinger equation, we will treat it as an ordinary field equation, for which we will first develop the Lagrange formalism. The Lagrangian density is:

$$\boxed{\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi}{2m} - V(\vec{x})\psi^*\psi} \tag{3.5}$$

Following the usual procedure, we obtain the momentum fields:

$$\Pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\psi^*; \quad \Pi_{\psi^*} = 0 \tag{3.6}$$

We could obtain our field equation by varying $\mathcal{L}$ with respect to the real and imaginary parts of ψ, but we can also vary with respect to ψ^* and ψ, since these are complex conjugates. This gives:

$$i\hbar\dot{\psi} + \frac{\vec{\nabla}\cdot\vec{\nabla}\psi}{2m} - V(\vec{x})\psi = 0 \tag{3.7}$$

which is the Schrödinger equation, as we had hoped. Now, our Hamiltonian density (as used in eq. (3.3)) is given by

$$\mathcal{H} = i\hbar\psi^*\dot{\psi} - \mathcal{L} = \hbar^2\frac{\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi}{2m} + V(\vec{x})\psi^*\psi \tag{3.8}$$

Here, ψ and $i\hbar\psi^*$ are canonical conjugates. In this equation, we could also introduce electromagnetic coupling.

Note that we could also have taken a more symmetric Lagrangian density, $\frac{i\hbar}{2}(\psi^*\dot{\psi} - \dot{\psi^*}\psi)$ to obtain this result.

3.1.3 Canonical quantization

The canonical quantization relation, where the ψ and ψ^* are promoted to ψ_{op} and $\psi^\dagger_{op}$, is given by

$$\boxed{[\psi_{op}(\vec{x},t), i\hbar\psi^\dagger_{op}(\vec{x}',t)] = i\hbar\delta^3(\vec{x}-\vec{x}')} \tag{3.9}$$

Note that the factor $i\hbar$ appears on both sides of the equation, and hence cancels. In terms of the $\mathbf{a}^{(\dagger)}_n$, we have

$$\sum_{n,n'}[\mathbf{a}_n, \mathbf{a}^\dagger_{n'}]\phi_n(\vec{x})\phi_{n'}(\vec{x}') = \delta^3(\vec{x}-\vec{x}') \tag{3.10}$$

where the generation and annihilation operators fulfill their usual commutation rule

$$[\mathbf{a}_n, \mathbf{a}^\dagger_{n'}] = \delta_{n,n'} \tag{3.11}$$

This gives the completeness relation: the $\phi_n(\vec{x})$ form a complete basis.

The Hamiltonian operator $\mathbf{H} = \int d^3x\mathcal{H}$ is, as discussed in chapter 1, given by

$$\mathbf{H} = \sum_n E_n\mathbf{a}^\dagger_n\mathbf{a}_n \tag{3.12}$$

Using eq. (3.11), we find the Heisenberg equations for the $\mathbf{a}_n$ and $\mathbf{a}^\dagger_n$:

$$\frac{d}{dt}\mathbf{a}_n(t) = \frac{i}{\hbar}[\mathbf{H}, \mathbf{a}_n] = -i\omega_n\mathbf{a}_n(t) \tag{3.13}$$

$$\frac{d}{dt}\mathbf{a}^\dagger_n(t) = \frac{-i}{\hbar}[\mathbf{H}, \mathbf{a}^\dagger_n] = i\omega_n\mathbf{a}^\dagger_n(t) \tag{3.14}$$

For the ψ_{op} and $\psi^\dagger_{op}$, we have:

$$i\hbar\frac{d}{dt}\psi_{op}(\vec{x},t) = -[\mathbf{H}, \psi_{op}(\vec{x},t)] \tag{3.15}$$

$$i\hbar\frac{d}{dt}\psi^\dagger_{op}(\vec{x},t) = [\mathbf{H}, \psi^\dagger_{op}(\vec{x},t)] \tag{3.16}$$

The energy states in Fock space are obtained as usually, by applying $\mathbf{a}^\dagger_n$ to the vacuum. The action of $\mathbf{a}_n$ and $\mathbf{a}^\dagger_n$ on a state is given by

$$\begin{aligned}\mathbf{a}_n\,|\ldots, N_n, \ldots\rangle &= \sqrt{N_n}\,|\ldots, N_n-1, \ldots\rangle \\ \mathbf{a}^\dagger_n\,|\ldots, N_n, \ldots\rangle &= \sqrt{N_n+1}\,|\ldots, N_n+1, \ldots\rangle\end{aligned}$$

Letting the $\psi^{(\dagger)}_{op}(\vec{x},t)$ act on the vacuum gives:

$$\psi_{op}(\vec{x},t)\,|0\rangle = 0, \tag{3.17}$$

since ψ_{op} contains only annihilators, and

$$\boxed{\psi^\dagger_{op}(\vec{x},t)\,|0\rangle = |\vec{X}\rangle_H} \tag{3.18}$$

which is a state with sharp position. The position $\vec{X}$ is fixed, but described by time-varying Hilbert space vectors, which is where the time dependence comes into the picture.

Exercise: prove eq. (3.18). Hint: use the Schrödinger equation for $\psi(\vec{x},t) = \langle X|\psi_H\rangle$, or write $\vec{X}_H(t) = \int d^3x'\psi^\dagger_{op}(\vec{x}',t)\vec{x}'\psi_{op}(\vec{x},t)$ and show that $\vec{X}_H|\vec{X}\rangle_H = \vec{X}\,|X\rangle_H$.

For the particle number density, we have:

$$\mathbf{n}(\vec{x},t) = \psi^\dagger_{op}(\vec{x},t)\psi_{op}(\vec{x},t) \tag{3.19}$$

$$\mathbf{n}(\vec{x}',t)\psi^\dagger_{op}(\vec{x}',t)\,|0\rangle = \delta(\vec{x}-\vec{x}')\underbrace{\psi^\dagger_{op}(\vec{x},t)\,|0\rangle}_{localized} \tag{3.20}$$

The total particle number N should be conserved, since we are dealing with normal QM systems, where particle production is forbidden. Indeed, using eq. (3.19) and the Heisenberg equation, we find that

$$\int d^3x\mathbf{n}(\vec{x},t) = N$$

is conserved.

Quite generally, for any Schrödinger operator $\mathbf{A}_S(\mathbf{X}_S,\mathbf{P}_S)$, we have an associated (Heisenberg) operator $\mathbf{A}_{F(ock)}$ acting on the Fock space of the one-particle state:

$$\boxed{\mathbf{A}_F(t) = \int d^3x\psi^\dagger_{op}(\vec{x},t)\mathbf{A}_S\psi_{op}(\vec{x},t)} \tag{3.21}$$

Using the commutator $[\psi_{op},\psi^\dagger_{op}] = \delta^3$, the fact that double annihilation on a one-particle state gives zero ($\psi_{op}\psi_{op}\,|1\text{ part.}\rangle = 0$), and the definition above, we see that commutation relations of the form

$$[\mathbf{A}_S,\mathbf{B}_S] = \mathbf{C}_S$$

translate to

$$[\mathbf{A}_F,\mathbf{B}_F] = \mathbf{C}_F$$

(the notation $\mathbf{A}_H$ is also used for $\mathbf{A}_F$).

An example of the above correspondence is the momentum operator, which generates translations. It is defined by $\mathbf{U}_{\vec{a}} = e^{-i\vec{\mathbf{P}}_F\cdot\vec{a}/\hbar}$, where now $\vec{\mathbf{P}}_F$ acts on states in the Fock space. Applying it to a state $|\vec{x}\rangle$ gives

$$\mathbf{U}_{\vec{a}} \underbrace{\psi^\dagger_{op}(\vec{x},t)\,|0\rangle}_{=|\vec{x}\rangle} = \mathbf{U}_{\vec{a}}\psi^\dagger_{op}(\vec{x},t)\mathbf{U}^\dagger_{\vec{a}} \underbrace{\mathbf{U}_{\vec{a}}\,|0\rangle}_{=|0\rangle}$$

Using the fact that $|0\rangle$ is invariant under translations, this becomes:

$$|\vec{x}+\vec{a}\rangle = \psi^\dagger_{op}(\vec{x}+\vec{a},t)\,|0\rangle$$

For infinitesimal translations, we have

$$\left[\vec{\mathbf{P}}_F, \psi^{(\dagger)}_{op}\right] = \frac{\hbar}{i}\vec{\nabla}\psi^{(\dagger)}_{op}$$

For rotations, generated by the angular momentum operator $\vec{\mathbf{J}}_F$, we have the same procedure: $\mathbf{U}_{\vec{\omega}} = e^{-i\vec{\mathbf{J}}_F\cdot\vec{\omega}/\hbar}$, where $\vec{\mathbf{J}}_F$ is given by

$$\vec{\mathbf{J}}_F = \int d^3x \psi^\dagger_{op}(\vec{x},t)\left(\vec{\mathbf{X}}_S \times \frac{\hbar}{i}\vec{\nabla}\right)\psi_{op}(\vec{x},t)$$

And again, for infinitesimal rotations:

$$\left[\vec{\mathbf{J}}_F, \psi^{(\dagger)}_{op}\right] = \left(\vec{\mathbf{X}}_S \times \frac{\hbar}{i}\vec{\nabla}\right)\psi^{(\dagger)}_{op}$$

3.2 Multiparticle Schrödinger equation

3.2.1 Bosonic multiparticle space

The description of systems consisting of more than one identical particles is a nice application of the new QFT-inspired formalism obtained from the second quantization. Note again that we are still talking about QM as we know it; only the language has changed.

Consider a system of N identical particles, which can be in states like

$$\left|n_{\vec{k}_1} n_{\vec{k}_2} \dots n_{\vec{k}_l}\right\rangle = \left|n_{\vec{k}_1}\right\rangle \otimes \left|n_{\vec{k}_2}\right\rangle \otimes \cdots \otimes \left|n_{\vec{k}_l}\right\rangle$$

where the possibility of polarization has been left out to avoid drowning in indices. The states $\left|n_{\vec{k}}\right\rangle$ are given by

$$\left|n_{\vec{k}}\right\rangle = \frac{(\mathbf{a}^\dagger_{\vec{k}})^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}\,|0\rangle \tag{3.22}$$

General states (still without polarization) look like this:

$$\sum_{l=1}^{\infty}\sum_{\{\vec{k}_1\dots\vec{k}_l\}} f(n_{\vec{k}_1}\vec{k}_1,\dots,n_{\vec{k}_l}\vec{k}_l)\left|n_{\vec{k}_1}\dots n_{\vec{k}_l}\right\rangle \tag{3.23}$$

Here, the order of the $\vec{k}_i$ is arbitrary, since the $\mathbf{a}^\dagger_{\vec{k}_i}$ commute. The total particle number is given by $N = \sum_{i=1}^{l} n_{\vec{k}_i}$. These states live in a Fock space obtained by a direct product of harmonic oscillator Hilbert spaces:

$$\mathfrak{F} = \prod_{l=1} \otimes \mathfrak{H}^{osc}_{\vec{k}_l}$$

They obviously inherit the linear structure of their constituents. Similarly, the inner product between two such states is obtained by taking the inner product in each $\mathfrak{H}_{\vec{k}}$ separately.

Another way of obtaining $\mathfrak{F}$ is summing spaces with fixed particle numbers:

$$\mathfrak{F} = \mathfrak{H}^{(0)} \oplus \mathfrak{H}^{(1)} \oplus \cdots \oplus \mathfrak{H}^{(l)} = \sum_{N} \oplus \mathfrak{H}^{(n)}$$

where

$$\mathfrak{H}^{(i)} = \prod_{i} \otimes \mathfrak{H}^{(1)} \quad \text{symmetrized}$$

is the Hilbert space for i identical particles.

$$\boxed{|\vec{x}_i \dots \vec{x}_l\rangle_H = N_s \psi^\dagger_{op}(\vec{x}_1, t) \dots \psi^\dagger_{op}(\vec{x}_l, t) \, |0\rangle} \tag{3.24}$$

The $\psi^\dagger_{op}$ generate *identical* particles, and hence commute. They are defined as follows:

$$\psi^\dagger_{op}(\vec{x}_j, t) = \begin{cases} \sum\limits_{\vec{k}} N_{\vec{k}} \mathbf{a}^\dagger_{\vec{k}}(t) e^{-i\vec{k}\cdot\vec{x}} & \text{for free particles} \\ \sum\limits_{\vec{k}} \mathbf{a}^\dagger_n(t) \phi^*_n(\vec{x}_j) \quad j = 1, \dots, l & \text{for bound particles} \end{cases} \tag{3.25}$$

Because of the fact that the $\psi^\dagger_{op}$ commute, we obtain $l!$ terms with the same set of $\vec{k}_i$ in the free particles-case. This gives rise to a factor of $l!$ in the inner product of a state with itself. This factor is always the same (exercise: show this), even if some of the $\vec{k}$ are identical. Examples are the case where all $\vec{k}$ are different and the case of m identical $\vec{k}$: in the first case, the norm squared of the state gets a factor $l!$, as just mentioned, and in the second, it is a factor

$$\begin{pmatrix} l \\ m \end{pmatrix} \cdot m!(l-m)! = l!$$

So, we set the normalization coefficient in eq. (3.24) $N_s = 1/\sqrt{l!}$.

Exercise: show, using the normalization coefficient derived above, that the following identity holds for a function f_{sym} that is symmetric in the x_l:

$$\int d^3x'_1 \dots d^3x'_l \, \langle \vec{x}_1 \dots \vec{x}_l | \vec{x}'_1 \dots \vec{x}'_l \rangle \, f_{sym}(\vec{x}'_1 \dots \vec{x}'_l) = f_{sym}(\vec{x}_1 \dots \vec{x}_l)$$

3.2.2 Interactions

The total Hamiltonian can be split up into the parts concerning the individual particles, parts resulting from interactions between two particles, etc.:

$$H = H^{(1)} + H^{(2)} + \dots \tag{3.26}$$

The parts $H^{(i)}$ are defined as follows:

$$H^{(1)} = \int d^3x \psi^\dagger_{op}(\vec{x},t)\left(\frac{-\hbar\nabla^2}{2m} + V(\vec{x})\right)\psi_{op}(\vec{x},t) \tag{3.27}$$

$$H^{(2)} = \int d^3x d^3x' \psi^\dagger_{op}(\vec{x},t)\psi^\dagger_{op}(\vec{x'},t)V(\vec{x},\vec{x'})\psi_{op}(\vec{x},t)\psi_{op}(\vec{x'},t) \tag{3.28}$$

The interaction potential $V(\vec{x},\vec{x}') = V^*(\vec{x},\vec{x}')$ is self-adjoint, e.g. $e^2/|\vec{x}-\vec{x}'|$ for the interaction between two electrons.

Exercise: using $[\psi_{op}(\vec{x},t), \psi^\dagger_{op}(\vec{x}',t)] = \delta^3(\vec{x}-\vec{x}')$, show that

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\underbrace{\psi(\vec{x}_1,\dots,\vec{x}_l,t)}_{=_H\langle\vec{x}_1\dots\vec{x}_l|\psi\rangle_H} &= i\hbar\frac{\partial}{\partial t}\langle 0|\frac{\psi_{op}(\vec{x}_1,t)\dots\psi_{op}(\vec{x}_l,1)}{\sqrt{l!}}|\psi\rangle_H \\
&= \langle 0| - [H^{(1)}+H^{(2)}, \frac{\psi_{op}(\vec{x}_1,t)\dots\psi_{op}(\vec{x}_l,t)}{\sqrt{l!}}|\psi\rangle \\
&\stackrel{!}{=} \langle 0|\sum_{i=1}^{l}\left(H^{(1)}_S(\vec{x}_i,\frac{\hbar}{i}\vec{\nabla}_i) + \frac{1}{2}\sum_{i,j=1,i\neq j}^{l} V(\vec{x}_i,\vec{x}_j)\right) \\
&\quad \times\frac{\psi_{op}(\vec{x}_1,t)\dots\psi_{op}(\vec{x}_1,t)}{\sqrt{l!}}|\psi\rangle \\
&= (H^{(1)}_S + H^{(2)}_S)\psi(\vec{x}_1,\dots,\vec{x}_l,t)
\end{aligned}$$

Given that we are dealing with two-body interactions here, instead of the single particles discussed so far, one could think that the commutation relation between ψ_{op} and $\psi^\dagger_{op}$ might not be the same as before. However, since the canonical momentum remains unchanged, we know that the old relation must still hold.

We started with the *linear* Schrödinger equation, but we could also have started with a nonlinear equation, such as the Gross-Pitaevsky equation for Bose-gasses. In this case, we would have to give up the superposition principle for probability amplitudes. Since we are doing nonrelativistic QM here, we prefer taking into account nonlinear parts only in $H^{(2)}$ and higher orders. Of course, one can also apply ordinary perturbative QFT, as will be discussed later on.

Remark

With the results obtained above, we can write any interaction completely in quantized fields. The electromagnetic interaction, with $\vec{A}(x)$, is, as we will see later on, obtained by minimal gauge coupling:

$$\frac{\hbar}{i}\vec{\nabla} \to \frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{\mathbf{A}}$$

$$H_0 = \int d^3x \psi^\dagger_{op}(\vec{x},t)\left\{\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 + V(\vec{x})\right\}\psi_{op}(\vec{x},t) + \frac{1}{8\pi}\int d^3x(\vec{E}^2+\vec{B}^2)$$

$$H_I = H^{(2)} + \int d^3x \psi^\dagger_{op}(\vec{x},t)\left[-\frac{e\hbar}{imc}\vec{\mathbf{A}}\cdot\vec{\nabla} + \frac{e^2}{2mc^2}\vec{\mathbf{A}}^2\right]\psi_{op}(\vec{x},t)$$

where ψ, $\psi^\dagger$ and $\vec{\mathbf{A}}$ are now quantum fields.

3.2.3 Fermions

We know from QM that we need the Pauli exclusion principle for electrons, protons, and other particles with half-integer spin. Later on, this will appear as a necessary consequence of QFT; we will see this in our discussion of the Dirac equation. Here, only the results will be given: fermionic fields are quantized by substituting anti-commutators, rather than commutators, for the Poisson bracket. We postulate:

$$[\mathbf{c}_n, \mathbf{c}^\dagger_{n'}]_+ = \mathbf{c}_n\mathbf{c}^\dagger_{n'} + \mathbf{c}^\dagger_{n'}\mathbf{c}_n = \delta_{nn'} \quad (3.29)$$

$$[\mathbf{c}_n, \mathbf{c}_{n'}]_+ = [\mathbf{c}^\dagger_n, \mathbf{c}^\dagger_{n'}]_+ = 0 \quad (3.30)$$

(the notation $\{\cdot,\cdot\}$ is often used as an alternative for $[\cdot,\cdot]_+$.)

Let us investigate the consequences of postulating anticommutating creation and annihilation operators. A striking result is that is immediately obvious is that $\mathbf{c}_n^2 = \mathbf{c}_n^{\dagger 2} = 0$. Now, consider the action of the occupation number operator $\mathbf{N} = \mathbf{c}^\dagger\mathbf{c}$, $\mathbf{N}|n\rangle = n|n\rangle$. Let it act on a state $\mathbf{c}|n\rangle$:

$$\begin{aligned}\mathbf{N}\mathbf{c}|n\rangle &= \mathbf{c}^\dagger \underbrace{\mathbf{c}\mathbf{c}}_{=0}|n\rangle = 0 = \\ &= (-\mathbf{c}\mathbf{c}^\dagger + 1)\mathbf{c}|n\rangle = (-\mathbf{c}\mathbf{N} + \mathbf{c})|n\rangle = (-n+1)\mathbf{c}|n\rangle = 0\end{aligned}$$

So, either $n = 1$ or $\mathbf{c}|n\rangle = 0$. Now, let it act on a state $\mathbf{c}^\dagger$:

$$\mathbf{N}\mathbf{c}^\dagger|n\rangle = \mathbf{c}^\dagger\mathbf{c}\mathbf{c}^\dagger|n\rangle = \underbrace{\mathbf{c}^\dagger(-\mathbf{c}^\dagger}_{=0}\mathbf{c} + 1)|n\rangle = \mathbf{c}^\dagger|n\rangle = (-n+1)\mathbf{c}^\dagger|n\rangle$$

So, either $n = 0$ or $\mathbf{c}^\dagger |n\rangle = 0$, giving the following relations:

$$\mathbf{c}^\dagger |1\rangle = 0, \quad \mathbf{c}|0\rangle = 0 \tag{3.31}$$
$$\mathbf{c}^\dagger |0\rangle = |1\rangle, \quad \mathbf{c}|1\rangle = |0\rangle \tag{3.32}$$

In other words, the only occupation numbers are 0 and 1, as we wanted for fermions. The normalization is as follows:

$$\langle 0|0\rangle = 1 \tag{3.33}$$
$$\langle 1|1\rangle = \langle 0|\mathbf{c}^\dagger\mathbf{c}|0\rangle = \langle 0|-\mathbf{c}\mathbf{c}^\dagger + 1|0\rangle = \langle 0|0\rangle = 1 \tag{3.34}$$

A general fermion state is:

$$\sum_{\{\vec{k}_1 \ldots \vec{k}_l\}} \prod_{i=1}^{l} f(\vec{k}_1 \ldots \vec{k}_l)\mathbf{c}^\dagger_{\vec{k}_i}|0\rangle \tag{3.35}$$

with $n_{\vec{k}_i} = 1$. Since the $\mathbf{c}^\dagger_{\vec{k}_i}$ anticommute, $\prod \mathbf{c}^\dagger_{\vec{k}_i}$ is totally antisymmetric, which means that only the totally antisymmetric part of f is interesting.

In x-space, states look like this:

$$|\vec{x}_1(t)\ldots\vec{x}_l(t)\rangle = \frac{1}{\sqrt{l!}}\psi^\dagger_{op}(\vec{x}_1,t)\ldots\psi^\dagger_{op}(\vec{x}_1,t)|0\rangle \tag{3.36}$$

where the $\psi^\dagger_{op}$ are, as usual, general solutions of the field equations:

$$\boxed{\psi^\dagger_{op}(\vec{x},t) = \sum_n \mathbf{c}^\dagger_n \phi^*_n(\vec{x})} \tag{3.37}$$

A state with l different oscillators occupied is given by:

$$\frac{1}{\sqrt{l!}}\sum_{\{1,\ldots,l\}_P}(-1)^P \phi^*_{k_{p_1}}(\vec{x}_1)\ldots\phi^*_{k_{p_l}}(\vec{x}_l)\mathbf{c}^\dagger_{k_1}\ldots\mathbf{c}^\dagger_{k_l}|0\rangle =$$
$$\frac{1}{\sqrt{l!}}\begin{vmatrix} \phi^*_{k_1}(\vec{x}_1) & \ldots & \phi^*_{k_l}(\vec{x}_1) \\ \phi^*_{k_1}(\vec{x}_2) & \ldots & \phi^*_{k_l}(\vec{x}_2) \\ \vdots & & \vdots \\ \phi^*_{k_1}(\vec{x}_l) & \ldots & \phi^*_{k_l}(\vec{x}_l) \end{vmatrix} \times \mathbf{c}^\dagger_{k_1}\ldots\mathbf{c}^\dagger_{k_l}|0\rangle \tag{3.38}$$

The determinant is called *Slater*-determinant, and gives rise to antisymmetric wave functions. It arises from the permutations which are summed over with changing sign (the part $\sum_{\{1,\ldots,l\}_P}(-1)^P$ in the left hand side).

Chapter 4

Quantizing covariant field equations

So far, our discussion of linear field equations has been quite generic, with the exception of the electromagnetic wave equation. The importance of this special case is indicated by the fact that it led Einstein to special relativity, and was the first triumph of QFT, which was absolutely essential for the acceptance of the theory. Therefore, it seems to make sense for us to inspect the relativistic aspects of the QFT quantization. Later on, a thorough classification of Lorentz covariant fields will be given; here we start with the simplest case, the Klein-Gordon equation. (Note that the wave equation is a special case of the Klein-Gordon equation: the one where $m = 0$.)

4.1 Lorentz transformations

We define *contravariant* and *covariant* spacetime vectors

$$x^\mu = (x^0, \vec{x}) \quad \text{contravariant} \tag{4.1}$$

$$x_\mu = g_{\mu\nu} x^\nu \quad \text{covariant} \tag{4.2}$$

with the metric tensor (in Cartesian coordinates)

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{4.3}$$

and $x^0 = ct$. Hereafter, we will use units where $c = 1$, so $x^0 = t$. Now, we set

$$x^2 = x^\mu x_\mu = (x^0)^2 - \vec{x}^2 = x^\mu g_{\mu\nu} x^\nu$$

This is invariant under *Lorentz transformations*:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{4.4}$$

for which the shorthand $x' = \Lambda x$ is frequently used. This invariance means that

$$\boxed{x^2 = x'^2} \tag{4.5}$$

which requires

$$\begin{aligned} x^{\mu} g_{\mu\nu} x^{\nu} = \Lambda^{\mu}{}_{\bar{\nu}} x^{\bar{\nu}} g_{\mu\nu} \Lambda^{\nu}{}_{\bar{\mu}} x^{\bar{\mu}} \Rightarrow \\ g_{\bar{\nu}\bar{\mu}} = \Lambda^{T\ \mu}_{\bar{\nu}} g_{\mu\nu} \Lambda^{\nu}{}_{\bar{\mu}} \end{aligned} \tag{4.6}$$

where we have relabelled $\mu \to \bar{\nu}$ and $\nu \to \bar{\mu}$ on the left hand side. In shorthand, we write

$$\boxed{g = \Lambda^T g \Lambda} \tag{4.7}$$

The gradient, $\partial/\partial x^{\mu}$, is a covariant vector if x^{μ} is a contravariant vector:

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} = (\Lambda^{-1})^{\nu}{}_{\mu} \frac{\partial}{\partial x^{\nu}} = (\Lambda^{-1})^{T}_{\ \mu\ \nu} \frac{\partial}{\partial x^{\nu}}$$

where we have used that $x' = \Lambda x$, $x = \Lambda^{-1} x'$, and hence $\partial x^{\nu}/\partial x'^{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu}$. So $\partial/\partial x^{\mu} = \partial_{\mu}$ transforms with $(\Lambda^{-1})^T$, i.e. like a covariant vector:

$$\begin{aligned} x'_{\mu} = g_{\mu\nu} x'^{\nu} &= g_{\mu\nu} \Lambda^{\nu}{}_{\lambda} x^{\lambda} \\ &= g_{\mu\nu} \Lambda^{\nu}{}_{\lambda} \underbrace{g^{\lambda\sigma}}_{=g_{\lambda\sigma}} x_{\sigma} \end{aligned}$$

which, using $g\Lambda g = (\Lambda^T)^{-1} = (\Lambda^{-1})^T$, gives

$$x'_{\mu} = (\Lambda^{-1})^{T}_{\ \mu\ \sigma} x_{\sigma}$$

4.2 Klein-Gordon equation for spin 0 bosons

4.2.1 Klein-Gordon equation

Let us begin by studying the simplest case of the Lorentz group: a scalar field, denoted by $\Phi(x)$. It can be transformed by a Lorentz transformation or a translation, giving the following general transformation:

$$\Phi(x) \to \Phi'(x) = \Phi(\Lambda^{-1}(x - a))$$

This is the active view of symmetry transformations. Above, we have used that

$$\boxed{\Phi'(x') = \Phi(x) \quad \text{with} \quad x' = \Lambda x + a} \tag{4.8}$$

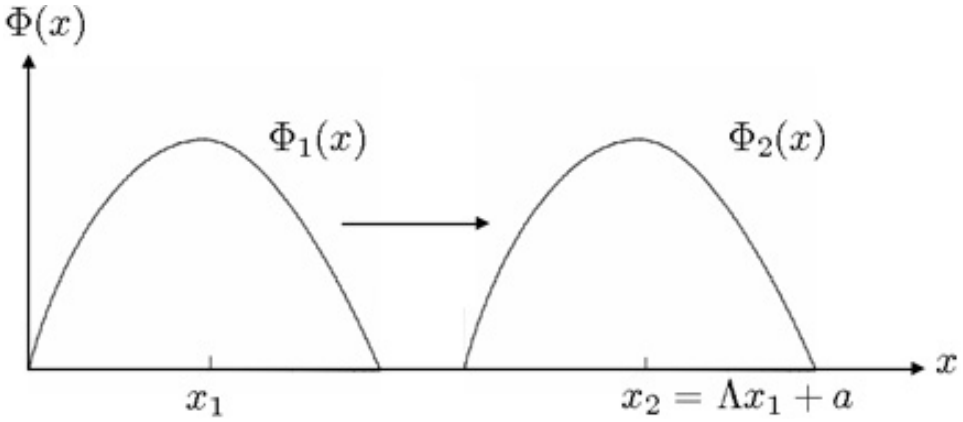


Figure 4.1: Transformation of $\Phi(x)$

We already know the Klein-Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\Phi(x) = 0 \tag{4.9}$$

(here written in units where $\hbar = c = 1$). In components, it looks like this:

$$\left[\left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^i}\right)^2 + m^2\right]\Phi(x^0, \vec{x}) = 0$$

(note that due to the squaring of the second term, the Einstein summing convention applies to the index i). It is *form invariant* under Lorentz transformations, since both m^2 and $\partial_\mu \partial^\mu$ transform like scalars, which are invariant under Lorentz transformations:

$$\begin{aligned} (\partial'_\mu \partial'^\mu + m^2)\Phi'(x') &= 0 \quad \rightarrow \quad (\text{rename } x \rightarrow x') \\ (\partial_\mu \partial^\mu + m^2)\Phi'(x) &= 0 \end{aligned}$$

Hence $\Phi'(x)$ is a solution of eq. (4.9) if $\Phi(x)$ is one.

4.2.2 Solving the Klein-Gordon equation in 4-momentum space

The general solution is obtained from a Fourier transform:

$$\Phi(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{\Phi}(k) e^{-ik_\mu x^\mu} \tag{4.10}$$

Plugging this into eq. (4.9) gives

$$(-k^2 + m^2)\tilde{\Phi}(k) = 0$$

so

$$\tilde{\Phi}(k) = a(k)2\pi\delta(k^2 - m^2)\theta(k_0) + a(k)2\pi\delta(k^2 - m^2)\theta(-k_0)$$

which we insert into eq. (4.10) after substituting $k \to -k$ in the second term ($a(-k) \to b^*(k)$):

$$\boxed{\Phi(x) = \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k_0)\left\{a(k)e^{-ikx} + b^*(k)e^{ikx}\right\}} \tag{4.11}$$

Here, $b^*(k)e^{ikx}$ represents the negative energy solutions.

For real fields, this means $b^*(k) = a^*(k)$, i.e. $a(-k) = a^*(k)$; see chapter 2. We can rewrite a complex field $\Phi(x)$ in terms of two real fields Φ_1 and Φ_2:

$$\begin{aligned} \Phi &= (\Phi_1 + i\Phi_2)/\sqrt{2} \\ \Phi^* &= (\Phi_1 - i\Phi_2)/\sqrt{2} \end{aligned}$$

yielding for $a(k)$ and $b(k)$:

$$a(k) = (a_1(k) + ia_2(k))/\sqrt{2} \tag{4.12}$$

$$b(k) = (a_1(k) - ia_2(k))/\sqrt{2} \tag{4.13}$$

Lorentz invariants

$\delta(k^2 - m^2)$ is Lorentz invariant: it makes sure the solution lies on the *mass shell*. $\theta(k_0)$ is Lorentz invariant on this mass shell (exercise: show this), and $\int d^4k$ is Lorentz invariant as well, since $k' = \Lambda k$, and $|\det \Lambda| = 1$, so $\int d^4k = \int d^4k'$. This implies that

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k_0) &= \int \frac{d^3k}{(2\pi)^3}\int dk_0\delta(k_0^2 - \vec{k}^2 - m^2)\theta(k_0) \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k_{0,+}} \qquad \text{where } k_{0,+} = +\sqrt{k_0} \end{aligned}$$

is Lorentz invariant, and, finally, that $\Phi(x)$ itself is a Lorentz scalar. To see this, rewrite eq. (4.11) as

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k_{0,+}}\left\{a(k)e^{-ikx} + b^*(k)e^{ikx}\right\}$$

and observe that the part between braces consists only of scalars. Later on, we will sometimes write $a(\vec{k})$ instead of $a(k)$, since k lies on the mass shell, and hence has only the degrees of freedom of the three-momentum.

The normalization of the $a(k)$ obtained above is somewhat different from the one used in chapter 2, since it has been adapted to the relativistic character of the equation. The a's are not exactly the ones from the harmonic oscillator case.

Reversing the equation, we can obtain Φ out of it:

$$\begin{aligned} a(\vec{k}) &= i\int d^3x e^{ikx}\overleftrightarrow{\partial_0}\Phi(x) \\ &:= i\int d^3x\left(-\frac{\partial e^{ikx}}{\partial x^0}\Phi(x)+e^{ikx}\frac{\partial\Phi(x)}{\partial x^0}\right) \\ b^*(\vec{k}) &= -i\int d^3x e^{-ikx}\overleftrightarrow{\partial_0}\Phi(x) \end{aligned}$$

Note that the inner product

$$(\Phi_1,\Phi_2) = i\int \Phi_1^*(x)\overleftrightarrow{\partial_0}\Phi_2(x)d^3x$$

is only positive definite for positive energy solutions, as the differentiation with respect to x^0 brings down a factor of $k_{0,+}$.

Exercise: show that $\theta(k_0)$ is Lorentz invariant. We also have $k_0' = \Lambda_a{}^0 k_0 + \Lambda_0{}^i k_i$. If k lies on the mass shell and $k_0 > 0$, we have the following inequalities:

$$\begin{aligned} |\vec{k}| &< k_0 \\ \sum_i(\Lambda_0{}^i)^2 &< (\Lambda_0{}^0)^2 \end{aligned}$$

writing $\Lambda^T g\Lambda = g$ in orthogonal coordinates (i.e. using the Minkowski metric). If $\Lambda_0{}^0 > 1$, we are dealing with the *orthochronous Lorentz group*, which we will encounter later on.

4.2.3 Quantization: canonical formalism

The Lagrange density corresponding to a real field $\Phi(x)$ with eq. (4.9) as 'equation of motion' is:

$$\boxed{\mathcal{L} = \frac{1}{2}\left(\partial_\mu\Phi(x)\partial^\mu\Phi(x) - m^2\Phi^2(x)\right)} \tag{4.14}$$

For a complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$, we have

$$\begin{aligned} \mathcal{L}_{com} &= \frac{1}{2}\sum_{j=1}^{2}\left\{\partial_\mu\Phi_j(x)\partial^\mu\Phi_j(x) - m^2\Phi_j^2(x)\right\} \\ &= \partial_\mu\Phi^*(x)\partial^\mu\Phi(x) - m^2\Phi^*(x)\Phi(x) \end{aligned}$$

Note that here, we use $x = (x^0,\vec{x})$ and $\partial_\mu\Phi\partial^\mu\Phi = \frac{\partial}{\partial x^0}\Phi\frac{\partial}{\partial x^0}\Phi - \frac{\partial}{\partial x^i}\Phi\frac{\partial}{\partial x^i}\Phi$.

Now, in the real case, we vary the Lagrangians with respect to Φ. In the complex case, we can choose between $\Phi_{1,2}$ and Φ, Φ^*, of which the latter

is the more practical one. This gives us the Euler-Lagrange equations of motion:

$$\boxed{-\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\frac{\partial \Phi}{\partial x^\mu})}\right)+\frac{\partial \mathcal{L}}{\partial \Phi}=0} \tag{4.15}$$

Note that the derivative $\frac{\partial}{\partial x^\mu}$ transforms like a *covariant* vector, whereas $\frac{\partial \mathcal{L}}{\partial(\frac{\partial \Phi}{\partial x^\mu})}$ is *contravariant*. This makes the first term a scalar, as it should be, since the second one is a scalar as well.

In the complex case, we obtain eq. (4.15) with Φ^* instead of Φ from varying with respect to Φ, and with Φ from varying with respect to Φ^*.

To obtain the Hamiltonian density, we need the momentum fields:

$$\Pi_\Phi(x)=\frac{\partial \mathcal{L}}{\partial(\frac{\partial \Phi}{\partial x^0})}=\frac{\partial \Phi}{\partial x^0} \tag{4.16}$$

This gives the Hamiltonian density

$$\mathcal{H}(x)=\Pi_\Phi\frac{\partial \Phi}{\partial x^0}-\mathcal{L}=\frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x^0}\right)^2+\left(\frac{\partial \Phi}{\partial x^i}\right)^2+m^2\Phi^2\right] \tag{4.17}$$

In the complex case, we have

$$\Pi_\Phi(x)=\frac{\partial \Phi^*}{\partial x^0}$$

$$\Pi_{\Phi^*}(x)=\frac{\partial \Phi}{\partial x^0}$$

and

$$\mathcal{H}_{com}(x)=\Pi_\Phi\frac{\partial \Phi}{\partial x^0}+\Pi_{\Phi^*}\frac{\partial \Phi^*}{\partial x^0}-\mathcal{L}=\Pi_\Phi\Pi_{\Phi^*}+\frac{\partial \Phi^*}{\partial x^i}\frac{\partial \Phi}{\partial x^i}+m^2\Phi^*\Phi \tag{4.18}$$

Both the complex and the real Hamiltonian densities are positive definite. From these, we obtain the Hamiltonian

$$H=\int d^3x\mathcal{H}(\Pi,\Phi) \tag{4.19}$$

Later on, we will see that this is the zero component of a four-vector.

The Poisson bracket of these fields is the usual delta function:

$$\left\{\Pi(\vec{x},t),\Phi(\vec{x}',t)\right\}_{\mathrm{P}}=\delta^3(\vec{x}-\vec{x}') \tag{4.20}$$

Quantizing this according to the usual rule gives the commutator

$$\left[\mathbf{\Pi}(\vec{x},t),\mathbf{\Phi}(\vec{x}',t)\right]=-i\delta^3(\vec{x}-\vec{x}') \tag{4.21}$$

where $\hbar = 1$. This does not look Lorentz covariant, since time and space are treated differently here. Later on, however, we will see that it *is* Lorentz covariant. In the complex case, we have the same commutator as above, or its Hermitian conjugate:

$$\left[\mathbf{\Pi}_{\Phi^\dagger}(\vec{x},t), \mathbf{\Phi}^\dagger(\vec{x}',t)\right] = -i\delta^3(\vec{x}-\vec{x}') \tag{4.22}$$

This implies:

$$\left[\frac{\partial \mathbf{\Phi}^\dagger(\vec{x},t)}{\partial t}, \mathbf{\Phi}(\vec{x}',t)\right] = -i\delta^3(\vec{x}-\vec{x}') = \left[\frac{\partial \mathbf{\Phi}(\vec{x},t)}{\partial t}, \mathbf{\Phi}^\dagger(\vec{x}',t)\right]$$

Rewriting the field $\Phi(x)$ as a Fourier transform, we get

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k_{0,+}}\left\{a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}\right\} \tag{4.23}$$

or, for the complex case,

$$\Phi_{com}(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k_{0,+}}\left\{a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}\right\} \tag{4.24}$$

Inserting this in the definition of H gives the following Hamiltonian for the real field:

$$H = \int d^3x\mathcal{H} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{4}\left\{a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right\} \tag{4.25}$$

Exercise: provide the intermediate steps. In the complex case, we have

$$\begin{aligned} H_{com} = &\int \frac{d^3k}{(2\pi)^3}\frac{1}{4}\left\{a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right\} + \\ &\int \frac{d^3k}{(2\pi)^3}\frac{1}{4}\left\{b^\dagger(\vec{k})b(\vec{k}) + b(\vec{k})b^\dagger(\vec{k})\right\} \end{aligned} \tag{4.26}$$

which features a second type of oscillators, the "b-particles".

Note that this integration measure is really the Lorentz covariant $\tilde{dk}$, followed by a factor of $k_{0,+}/2$, which we know from the traditional Hamiltonian (it was called $\omega/2$ before):

$$H = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k_{0,+}}\frac{k_{0,+}}{2}(\dots) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{4}(\dots)$$

When quantizing this field, we promoted the a and a^* to operators $\mathbf{a}$ and $\mathbf{a}^\dagger$, and similarly for the b in the complex case. The $\mathbf{a}$, $\mathbf{a}^\dagger$ have the following commutation relation:

$$[\mathbf{a}(\vec{k}), \mathbf{a}^\dagger(\vec{k}')] = 2k_{0,+}(2\pi)^3\delta^3(\vec{k}-\vec{k}') \tag{4.27}$$

The same holds for the **b**'s. Also, any **a** commutes with any **b**, i.e. they are completely independent and on the same level.

Note: in the literature, the factor of $2k_{0,+}$ is sometimes omitted; the notation employed here, which does include it, along with some π-factors, is called the "*Cambridge notation*".

Both the **a**, $\mathbf{a}^\dagger$ and the **b**, $\mathbf{b}^\dagger$ build up a symmetric Fock space. Mixed **a**, **b** combinations therefore act in a product space. A state with n a-particles and n' b-particles looks like this:

$$\prod_{i=1}^{n+n'} \frac{d^3k_i}{(2\pi)^3 2k_{i,0}} \frac{\tilde{c}(k_1 \dots k_n)}{\sqrt{n!}} \frac{\tilde{\tilde{c}}(k_{n+1} \dots k_{n'})}{\sqrt{n'!}} \times \\ \mathbf{a}^\dagger(\vec{k}_1) \dots \mathbf{a}^\dagger(\vec{k}_n) \mathbf{b}^\dagger(\vec{k}_{n+1}) \dots \mathbf{b}^\dagger(\vec{k}_{n+n'}) \, |0\rangle \tag{4.28}$$

Since these states have to be normalized, the normalization functions $\tilde{c}$ and $\tilde{\tilde{c}}$ have to obey:

$$\langle\psi|\psi\rangle = 1 = \int \frac{d^3k_1}{(2\pi)^3 2k_{1,0}} \cdots \frac{d^3k_{n+n'}}{(2\pi)^3 2k_{n+n',0}} |\tilde{c}|^2 |\tilde{\tilde{c}}|^2 = 1 \tag{4.29}$$

4.2.4 Charge conjugation

Consider the Hermitian conjugate of a complex quantized field $\Phi(x)$:

$$\Phi^\dagger(x) = \int \left\{ \mathbf{a}^\dagger(\vec{k}) e^{ikx} + \mathbf{b}(\vec{k}) e^{-ikx} \right\}$$

The roles of **a** and **b** have clearly been exchanged. Hence, we can conclude that the b-particles are antiparticles of the a-particles. This is illustrated more clearly by the coupling of a complex ("π-meson") field $\Phi(x)$ to an electromagnetic field:

$$\partial_\mu \to \partial_\mu + ieA_\mu = \mathcal{D}_\mu$$

Coupling a charged field to a vector potential, one replaces the momentum $p_\mu \leftrightarrow i\partial_\mu$ by the kinetic momentum in the Hamiltonian $p_\mu - eA_\mu$; this is called "minimal coupling" and leads to the modified Klein-Gordon equation:

$$\boxed{(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\Phi(x) = 0} \tag{4.30}$$

which is *gauge invariant* under the following transformations:

$$\Phi \to \Phi'(x) = e^{-ie\Lambda(x)} \Phi(x)$$
$$A_\mu(x) \to A'_\mu(x) = A_\mu + \partial_\mu \Lambda$$

since the "covariant derivative" transforms as

$$\mathcal{D}'_\mu \Phi'(x) = (\partial_\mu + ieA_\mu + ie\partial_\mu\Lambda) e^{-ie\Lambda(x)} \Phi(x) \\ = e^{-ie\Lambda(x)} \mathcal{D}_\mu \Phi(x) \tag{4.31}$$

(∂_μ acting on $e^{-ie\Lambda(x)}$ gives a term $-ie\partial_\mu\Lambda$ in the factor between brackets).

Φ^* fulfills the same equation with $\boxed{e \to -e}$, which is the same as replacing $\mathcal{D}_\mu$ with $\mathcal{D}_\mu^*$. Since changing Φ into Φ^* means exchanging $\mathbf{a}$ and $\mathbf{b}$, this exchange really achieves charge conjugation. Now, we would like to have a charge conjugation operator, in order to make this operation mathematically more fitting to the model. This operator should be unitary: reversing a charge conjugation should give the original situation. Its action should of course be:

$$\Phi^\dagger(x) = \mathbf{U}_C \Phi(x) \mathbf{U}_C^{-1} \qquad (\mathbf{U}_C^{-1} = \mathbf{U}_C^\dagger) \tag{4.32}$$

(this is equivalent to $\mathbf{b}(\vec{k}) = \mathbf{U}_C \mathbf{a}(\vec{k}) \mathbf{U}_C^\dagger$). Exercise: show that this operator is given by

$$\mathbf{U}_C = \exp\left(-i\pi \int d\tilde{k} \frac{1}{2} \left\{ \mathbf{a}^\dagger \mathbf{a} + \mathbf{b}^\dagger \mathbf{b} - \mathbf{a}^\dagger \mathbf{b} - \mathbf{b}^\dagger \mathbf{a} \right\}\right)$$

Hint: use

$$\Phi_2 = \frac{\Phi - \Phi^*}{i\sqrt{2}} = \int d\tilde{k} \left(\frac{\mathbf{a} - \mathbf{b}}{i\sqrt{2}} e^{i\vec{k}\cdot\vec{x}} - \frac{\mathbf{a}^\dagger - \mathbf{b}^\dagger}{i\sqrt{2}} e^{-i\vec{k}\cdot\vec{x}} \right)$$

Then, rewrite the integrand in the formula for $\mathbf{U}_C$ as $\mathbf{a}_2^\dagger(\vec{k}) \mathbf{a}_2(\vec{k})$, where $\mathbf{a}_2 := \frac{\mathbf{a}-\mathbf{b}}{i\sqrt{2}}$.

4.3 Microcausality

In its quantized form, the Klein-Gordon equation still should fulfill the requirements of special relativity. In evaluating whether it does, we will use complex fields; we could also have used real ones.

The commutator of a field $\Phi(x)$ at position x with its Hermitian conjugate at position y is given by

$$\begin{aligned}
&[\Phi(x), \Phi^\dagger(y)] =: i\Delta(x-y) \\
&= \int \frac{d^3k}{2k_0(2\pi)^3} \frac{d^3k'}{2k_0(2\pi)^3} \left([\mathbf{a}(\vec{k}), \mathbf{a}^\dagger(\vec{k}')] e^{-ikx+ik'y} + [\mathbf{b}(\vec{k}), \mathbf{b}^\dagger(\vec{k}')] e^{ikx-ik'y} \right) \\
&= \int \frac{d^3k}{2k_0(2\pi)^3} \left(e^{-ik(x-y)} - e^{ik(x-y)} \right)
\end{aligned} \tag{4.33}$$

Renaming $\vec{k} \to -\vec{k}$ in the second term, the integrand can be rewritten as $-2ie^{i\vec{k}\cdot(\vec{x}-\vec{y})} \sin(k_0(x^0 - y^0))$. This implies that if $x^0 = y^0$, $\Delta = 0$. Now, if the difference between x and y is spacelike, one can always transform to a frame of reference where $(x-y)^0 = 0$. Since the commutator is invariant under transformations, this means that for a spacelike pair of points, Δ is always zero. Thus, *microcausality* is conserved. Commutators of Φ or $\Phi^\dagger$ alone vanish identically.

4.4 Nonrelativistic limit

We have the Klein-Gordon equation:

$$(\partial_0^2 - \vec{\partial}^2 + m^2)\Phi(\vec{x}, t) = 0 \tag{4.34}$$

Extracting the time derivative and expanding around m gives:

$$\begin{aligned} i\partial_0\Phi(\vec{x}, t) &= \pm(m^2 - \vec{\partial}^2)^{1/2}\Phi(\vec{x}, t) \\ &= \pm\left(m - \frac{1}{2m}\vec{\partial}^2 + \dots\right)\Phi(\vec{x}, t) \end{aligned}$$

Retaining only the first two terms, and choosing the positive sign convention, we obtain the non-relativistic Schrödinger equation with a constant potential (resulting from the first term, m). Note that this is only valid in the non-relativistic limit, where $|\vec{k}| << m$: the full operator $(m^2 - \vec{\partial}^2)^{1/2}$ is nonlocal.

The inner product of a field with itself is given by:

$$(\Phi, \Phi) = \int d^3x \Phi^*(x) \overleftrightarrow{\partial}_0 \Phi(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \left(a(\vec{k})a^*(\vec{k}) - b^*(\vec{k})b(\vec{k})\right) \tag{4.35}$$

which is *not* positive definite. We can interpret $|a(\vec{k})|^2 \frac{d^3k}{(2\pi)^3 2k_0}$ as a probability density if we consider only solutions with positive energy, i.e. where $b \equiv 0$. Negative energy solutions would have disastrous results in interaction theory.

In QM, the normalization factor of $1/(2\pi)^3 2k_{0,+}$ is absent, and we just have $|\tilde{\Psi}(\vec{k})|^2 d^3k$. Fourier transforming this gives the wave function $\Psi(\vec{x}, t)$, which has a positive definite inner product for positive energy solutions.

Taking a localised $\Psi_{\vec{x}_0}(\vec{x}, 0) = \delta^3(\vec{x} - \vec{x}_0)$, Fourier transforming back to momentum space ($\Rightarrow a(\vec{k})$) we obtain $\Phi_{\vec{x}_0}(\vec{x}, 0)$, $\Phi_{\vec{x}_0}(\vec{x}, 0) \sim \left(\frac{m}{r}\right)^{5/4} H_{5/4}(imr)$ with $r = |\vec{x} - \vec{x}_0|$. Although this drops exponentially for $r >> m^{-1}$, it is not really localized.

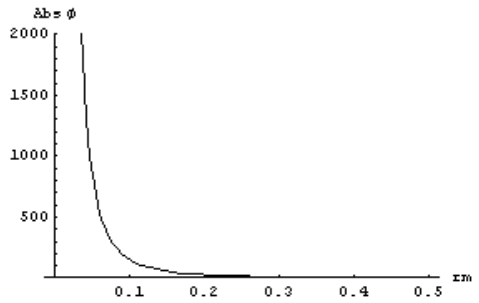


Figure 4.2: Unsharp position

The position operator

In position space, the position operator $\vec{\mathbf{X}}$ becomes $\vec{x}_0$:

$$\vec{\mathbf{X}}\Phi_{\vec{x}_0}(\vec{x},0) = \vec{x}_0\Phi_{\vec{x}_0}(\vec{x},0)$$

In momentum space, we get something different from QM:

$$\vec{\mathbf{X}} \to i\vec{\nabla}_{\vec{p}} - \frac{i\vec{p}}{2p_0^2}$$

The extra term of $-i\vec{p}/2p_0^2$ is responsible for the *zitterbewegung*, which obscures the exact position.

Chapter 5

Interacting fields, S-matrix, LSZ reduction

5.1 Non-linear field equations

Last chapter, we studied the Klein-Gordon equation, $(\partial^2 + m^2)\Phi(x) = 0$. Now, let us consider *non-linear* equations, such as:

$$(\partial^2 + m^2)\Phi(x) = \frac{\lambda}{3!}\Phi^3 \tag{5.1}$$

where we have a (local) self interaction of the field, which is obtained from a term $\mathcal{L}_{int}$ in $\mathcal{L}$: $\frac{-\lambda}{4!}\Phi^4$. Sometimes it is convenient to separate the nonlinear effects, introducing a source field $j(x)$ in the above equation::

$$(\partial^2 + m^2)\Phi(x) = j(x) \tag{5.2}$$

and obtain solutions to eq. (5.1) in this way.

In eq. (5.1), the canonical momentum field is unchanged, since $\mathcal{L}_{int}$ does not contain any derivatives, and therefore the canonical commutation relations are still valid. However, the Fourier decomposition used before is not available here, because the equation is, as said, non-linear.

Remark

It is possible to construct the Noether currents and charges (the generators of transformations) and check whether Lorentz covariance is consistent with the canonical commutation relations. This turns out to be the case, as we will see later on.

In this course, we will mostly treat $\mathcal{L}_{int}$ as a small perturbation of the free theory, which brings us to perturbation theory. First, however, we will present a more general framework: the *LSZ formalism*, named after its inventors Lehmann, Szymanzik and Zimmermann.

5.1.1 Relation to observation

Elementary particle physics

In elementary particle physics, we study the *scattering* of particles. As we will see, this problem can be reduced to calculating vacuum expectation values (v.e.v.'s) in the ground states of the interacting theory:

$$\langle 0|\, T(\Phi(x_1,t_1)\dots\Phi(x_n,t_n))\,|0\rangle$$

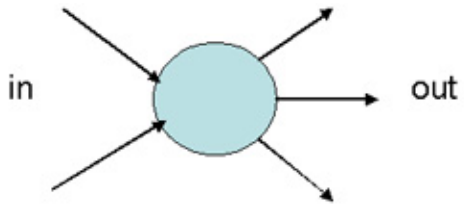
Statistical mechanics

Although QFT is most directly connected to elementary particle physics, there also are links to other areas, such as statistical mechanics. There, one works with phase space quantities, whose dynamics are described by a Hamiltonian. In the QM version, one has a Hilbert space, in which states, including multiparticle states, live; for quantum gases we have a Fock space. In both the Hilbert and the Fock space, observables correspond to operators, as is known from QM. Quantities that are averaged over space, such as magnetisation $M(\vec{x},t)$, correspond to field operators $\Phi(\vec{x},t)$. Then, it is possible to investigate correlations in some ensemble

$$\langle\Phi(x_1,t_1)\dots\Phi(x_n,t_n)\rangle$$

which, in the canonical ensemble, are weighed by a factor of e^{-H/k_BT} for a temperature T. We will come back to this case later, in the context of path integrals.

5.2 Interaction of particles in QFT



In the QFT description of interacting particles, the "in" and "out" fields are free fields approached asymptotically by the interacting fields. The "in" and "out" Fock spaces with state vectors $|\text{in}\rangle$ and $|\text{out}\rangle$, and the quantum fields $\Phi_{in}(x)$ and $\Phi_{out}(x)$, are *unitarily equivalent*: physically equal states $|\text{in}\rangle$, $|\text{out}\rangle$ are described by *different* state vectors. In the Heisenberg picture (as usual in QFT), we have

$$\boxed{\Phi_{out}(x) = S^{-1}\Phi_{in}(x)S} \quad \text{and} \tag{5.3}$$

$$|\text{out}\rangle = S^{-1}\,|\text{in}\rangle = S^{\dagger}\,|\text{in}\rangle \tag{5.4}$$

with a unitary matrix S. Note that states containing less than two particles do not interact, as there is nothing to interact with:

$$|0\rangle_{in} = |0\rangle_{out} \qquad \text{and} \qquad |1\rangle_{in} = |1\rangle_{out}$$

Now consider the limit where $\Phi(x)$ approaches $\Phi_{in}(x)$. The equation

$$(\partial^2 + m^2)\Phi(x) = j(x)$$

is solved by

$$\boxed{\Phi(x) = \Phi_{hom}(x) + \int d^4y\, G_{ret}(x-y,m)j(y)} \tag{5.5}$$

where G_{ret} is the *retarded Green's function*, for which we have

$$\begin{aligned}(\partial^2 + m^2)G_{ret}(x-y,m) &= \delta^4(x-y)\\ \text{and} \quad G_{ret}(x-y,m) &= 0 \quad \text{for } x_0 < y_0\end{aligned}$$

since the current $j(x)$ influences only the future. From this equation, one might naïvely assume that $\Phi(x)$ approaches the homogeneous solution as $x_0 \to -\infty$, as mentioned at the beginning of this section. However, before being able to assume this, one has to consider the covergence type. The problem is that the normalizations of $\Phi(x)$ and $\Phi_{hom}(x)$ are not generally the same:

$$\begin{aligned}\langle 1\vec{p}|\, \Phi_{in}(x)\, |0\rangle &\neq \langle 1\vec{p}|\, \Phi(x)\, |0\rangle\\ \Phi_{in}(x) &= e^{i\mathbf{P}x}\Phi_{in}(0)e^{-i\mathbf{P}x}\\ \Phi(x) &= e^{i\mathbf{P}x}\Phi(0)e^{-i\mathbf{P}x}\end{aligned}$$

Both matrix elements are proportional to $e^{i\mathbf{P}x}$, but the norms are not equal, since $\Phi(x)\,|0\rangle$ is in general not a pure one-particle state. States in Hilbert or Fock space are normalized by their inner product with themselves, whereas operators are normalized through commutators. So, what we have to consider is the convergence of $\Phi(x)$ towards $\Phi_{in}(x)$ multiplied by some scalar:

$$\lim_{x_0\to-\infty} \Phi(x) = \underbrace{Z^{1/2}\Phi_{in}(x)}_{\text{hom. solution}} \tag{5.6}$$

5.2.1 The Yang-Feldman equations

This idea has been implemented in the Yang-Feldman equations:

$$\boxed{\sqrt{Z}\Phi_{in}(x) = \Phi(x) - \int d^4x G_{ret}(x-y,m)j(y)} \tag{5.7}$$

$$\boxed{\sqrt{Z}\Phi_{out}(x) = \Phi(x) - \int d^4x G_{av}(x-y,m)j(y)} \tag{5.8}$$

These can be used as definitions for $\Phi_{\text{in,out}}(x)$, since $\sqrt{Z}$ is fixed by the canonical commutation relation for $\Phi_{\text{in,out}}$. It also is the same in both equations. To see this, consider $\langle \text{1-part.}| \Phi(x) |0\rangle$, and note that in this case, the second term does not contribute.

We do not have convergence in the operator sense (or *strong convergence*) for $\Phi(x)$: it is not true that $\Phi(x) \to Z^{1/2}\Phi_{in}(x)$, since this is not compatible with the canonical commutation relations. Assuming $\Phi(x) \to Z^{1/2}\Phi_{in}(x)$, the equal time commutator would give:

$$\begin{aligned} [\partial_0\Phi(x), \Phi(y)]_{x_0=y_0} &= -i\delta^3(\vec{x}-\vec{y}) \qquad \text{so} \\ Z[\partial_0\Phi_{in}(x), \Phi(y)]_{x_0=y_0} &= -i\delta^3(\vec{x}-\vec{y}) \qquad \text{for } x_0 \to -\infty \end{aligned}$$

which cannot be true, since Z is not in general equal to 1. Therefore, we only have convergence in the following sense:

- Only individual matrix elements converge. Since not all matrix elements necessarily converge at the same rate, the whole matrix, or the operator, does not converge.
- Wave packets built out of matrix elements, such as

$$\Phi(f,t) = i\int d^3x f^*(x) \overleftrightarrow{\partial}_0 \Phi(\vec{x},t)$$

  with $(\partial^2 + m^2)f = 0$, also converge.

This type of convergence is also called *weak convergence*.

Remark

$j(\vec{x},t) \to 0$ for $t \to \pm\infty$ in the operator sense is *not* possible, unless one goes to the Bogoliubov-Shirkov construction, where couplings are made to go to zero as $t \to \infty$ (see their textbook on QFT).

5.3 Green's functions

We have already seen the *causal Green's function*,

$$\begin{aligned} D_C(x'-x) &= i\Delta(x'-x) = [\Phi_{free}(x'), \Phi_{free}(x)] \\ &= \langle 0| [\Phi_{free}(x'), \Phi_{free}(x)] |0\rangle \end{aligned}$$

which is given by

$$D_C(x'-x) = \int \frac{d^3k}{2k_0(2\pi)^3} \left(e^{-ik(x'-x)} - e^{ik(x'-x)}\right) \tag{5.9}$$

and is zero for $(x'-x)^2 < 0$, i.e. for spacelike displacements. **Exercise**: show that D_C can be written as follows with the help of the zeroth-order Bessel function J_0:

$$D_C(x) = D_C(x^0, \vec{r}) = \frac{i}{4\pi r}\frac{\partial}{\partial r}\begin{cases} J_0(m((x^0)^2 - r^2)^{1/2}) & (x^0)^2 > r^2 \\ 0 & (x^0)^2 < r^2 \end{cases}$$

The canonical commutation relation ($[\partial_t \Phi(x'), \Phi(x')]_{ET} = -i\delta^3(\vec{x}' - \vec{x})$) gives

$$\frac{\partial}{\partial x'^0} D_C(x'-x) = -i\delta^3(\vec{x}' - \vec{x}) \tag{5.10}$$

Now, we can see that the *retarded* and *advanced* Green's functions from electrodynamics are given by:

$$\boxed{D_R(x'-x) = \theta(x'^0 - x^0) D_C(x'-x)} \tag{5.11}$$

$$\boxed{D_A(x'-x) = \theta(x^0 - x'^0) D_C(x'-x)} \tag{5.12}$$

Exercise: check that $(\partial^2_{x'} + m^2) D_{R,A}(x'-x) = -i\delta^4(x'-x)$. Use $\frac{\partial}{\partial x'_0}\theta(x'^0 - x^0) = \delta(x'^0 - x^0)$ and the fact that δ' is defined by partial integration.

The usual way to obtain $D_{R,A}$ is solving the differential equation via a Fourier transform and realizing the boundary conditions by a choice at the poles of the integrand:

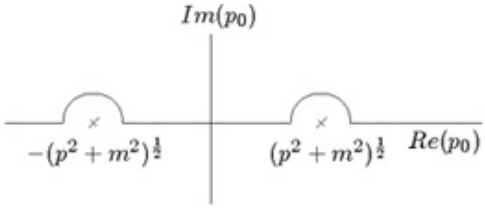


Figure 5.1: Integration path for D_R

$$\begin{aligned} D_R(x) &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2} e^{-ipx} \\ &= -\int \frac{d^3p}{(2\pi)^3}\int \frac{dp_0}{2\pi i}\frac{1}{p_0^2 - \vec{p}^2 - m^2} e^{-ip_0 x^0} e^{i\vec{p}\cdot\vec{x}} \end{aligned} \tag{5.13}$$

For $x^0 > 0$, we close the integration loop in the lower half plane, and for $x^0 < 0$ in the upper half plane. For D_A, we pass by the poles in the lower half plane and close the loop in the upper half plane. The denominator in the integrand comes from the two poles:

$$\frac{1}{p_0 + (\vec{p}^2 + m^2)^{1/2} + i\epsilon} \times \frac{1}{p_0 - (\vec{p}^2 + m^2)^{1/2} + i\epsilon} \sim \frac{1}{p_0^2 - \vec{p}^2 - m^2 + i\epsilon}$$

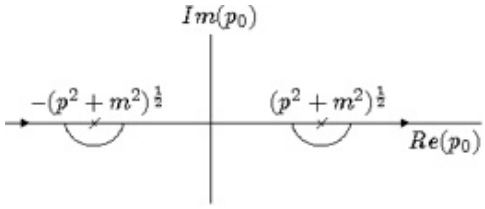


Figure 5.2: Integration path for D_A

Later on, the *Feynman propagator* will be important; it is obtained by the following integration path, and is given by

$$D_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ipx} \tag{5.14}$$

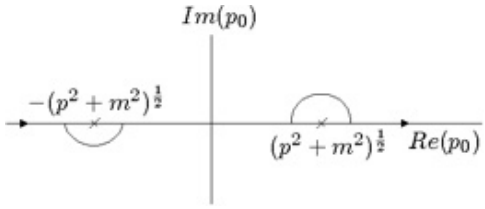


Figure 5.3: Integration path for D_F

For $x^0 > 0$, the integral loop is closed in the lower half plane, so this pole contributes

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_{0,+}} e^{-ipx} \Big|_{p_0 = p_{0,+}}$$

which is the first half of the commutator from D_C; for $x^0 < 0$, the loop is closed in the upper half plane, which gives the second half:

$$-\int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_{0,+}} e^{-ipx} \Big|_{p_0 = -p_{0,+}}$$

Altogether, this gives

$$D_F(x - x') = \theta(x'^0 - x^0) \langle 0| \Phi(x')\Phi(x) |0\rangle + \theta(x^0 - x'^0) \langle 0| \Phi(x)\Phi(x') |0\rangle$$

or

$$\boxed{D_F(x - x') = \langle 0| T(\Phi(x')\Phi(x)) |0\rangle} \tag{5.15}$$

5.4 Spectral representation of commutator vev

Consider the vacuum expectation value of the commutator of a field Φ at two positions x and y, $\langle 0|\,[\Phi(x),\Phi(y)]\,|0\rangle$. With a complete set of states $|\alpha\rangle$, it can be rewritten:

$$\langle 0|\,[\Phi(x),\Phi(y)]\,|0\rangle = \sum_\alpha \Big(\langle 0|\,\Phi(0)\,|\alpha\rangle\, e^{ip_\alpha(x-y)}\, \langle\alpha|\,\Phi(0)\,|0\rangle - \langle 0|\,\Phi(0)\,|\alpha\rangle\, e^{ip_\alpha(y-x)}\, \langle\alpha|\,\Phi(0)\,|0\rangle \Big)$$

(remember $\Phi(x) = e^{iPx}\Phi(0)e^{-iPx}$ and $\mathbf{P}\,|0\rangle = 0$). Defining

$$\rho(k) = (2\pi)^4 \sum_\alpha \delta^4(k-p_\alpha)|\,\langle 0|\,\Phi(x)\,|\alpha\rangle\,|^2 = \sigma(k^2)\theta(k_0) \tag{5.16}$$

and using $1 = \int d^4k\delta^4(k-p_\alpha)$, we can rewrite this as

$$\begin{aligned}\langle 0|\,[\Phi(x),\Phi(y)]\,|0\rangle &= \int \frac{d^4k}{(2\pi)^4}\rho(k)(e^{-ik(x-y)} - e^{ik(x-y)}) \\ &= \int_0^\infty dm^2\sigma(m^2)D_C(x-y,m^2)\end{aligned} \tag{5.17}$$

which is called the *spectral representation*. Taking out physical 1-particle contributions with the weak asymptotic condition allows us to rewrite this as

$$ZD_C(x-y,m^2) + \int_{>m^2}^\infty dm'^2\sigma(m'^2)D_C(x-y,m'^2)$$

Now applying the operator $-i\partial_0$ to both sides and setting $x^0 = y^0$, we get a factor of $\delta^3(\vec{x}-\vec{y})$ on both sides. Setting the residues equal to each other gives the *Källen-Lehmann sum rule*:

$$\boxed{1 = Z + \int_{>m^2}^\infty dm'^2\sigma(m'^2)} \tag{5.18}$$

From this we can conclude that $0 \le Z \le 1$. This will not be obvious in perturbation theory, where $Z^{-1} = 1 + g^2 \times \ldots$ is divergent.

5.5 LSZ reduction formalism

The LSZ reduction formalism will be particularly useful for abstract S-matrix theory; in the Lagrangian formulation and in perturbation theory one finally returns to the interaction picture.

The idea is to take a one-particle state out of the incoming state, calculate its contribution, and thus reduce the problem at hand to a $n-1$-particle

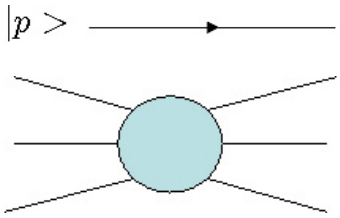


Figure 5.4: Disconnected state running through

state.

$$\begin{aligned} S_{\beta,\alpha p} &= \langle\beta_{out}|(\alpha p)_{in}\rangle = \langle\beta_{out}|\,\mathbf{a}^\dagger_{in}(p)\,|\alpha_{in}\rangle \\ &= \langle\beta_{out}|\,\mathbf{a}^\dagger_{out}(p)\,|\alpha_{in}\rangle - \qquad \text{(I)} \\ &\quad i\,\langle\beta_{out}|\int d^3x e^{-ipx}\overleftrightarrow{\partial_0}(\Phi_{in}(x)-\Phi_{out}(x))\,|\alpha_{in}\rangle \qquad \text{(II)} \end{aligned}$$

where in the second term, we have inverted the formula for Φ to rewrite $\mathbf{a}_{in} - \mathbf{a}_{out}$ in terms of Φ. In term (I), $\langle p|$ is being annihilated from the outgoing state, so it only contributes in case of a state $|p\rangle$ running through but not interacting. Since we are interested in the interaction, we will neglect this term in the end.

For term (II), we now use the (weak) convergence of $\Phi_{in/out}(x^0,\vec{x})$ to $1/\sqrt{Z}\Phi(x^0,\vec{x})$ for $x^0 \to \pm\infty$; plus for outgoing and minus for incoming. This allows us to rewrite it as

$$\begin{aligned} \text{(II)} = &\lim_{x^0\to\infty}\frac{i}{\sqrt{Z}}\int d^3x e^{-ipx}\overleftrightarrow{\partial_0}\,\langle\beta_{out}|\,\Phi(x^0,\vec{x})\,|\alpha_{in}\rangle - \\ &\lim_{x^0\to-\infty}\frac{i}{\sqrt{Z}}\int d^3x e^{-ipx}\overleftrightarrow{\partial_0}\,\langle\beta_{out}|\,\Phi(x^0,\vec{x})\,|\alpha_{in}\rangle \\ = &\frac{i}{\sqrt{Z}}\int_{-\infty}^{\infty}dx^0\frac{\partial}{\partial x^0}\int d^3x e^{-ipx}\overleftrightarrow{\partial_0}\,\langle\beta_{out}|\,\Phi(x^0,\vec{x})\,|\alpha_{in}\rangle \end{aligned}$$

With $(\partial_0^2 - \vec{\partial}^2 + m^2)e^{-ipx} = 0$, the mixed term resulting from ∂_0^2 cancels, so we get

$$\text{(II)} = \frac{i}{\sqrt{Z}}\int d^4x e^{-ipx}(\partial^2+m^2)\,\langle\beta_{out}|\,\Phi(x)\,|\alpha_{in}\rangle$$

where the Klein-Gordon operator acting on $\Phi(x)$ gives $j(x)$.

For outging particles ($\langle\beta_{out}| = \langle(\gamma p')_{out}|$), the story is more or less the same:

$$\begin{aligned} \left\langle(\gamma p')_{out}\right| &= \langle\gamma_{out}|\,\Phi(x)\left|(\alpha\setminus p')_{in}\right\rangle \qquad \text{(I)} \\ &\quad + \langle\gamma_{out}|\,\mathbf{a}_{out}(p')\Phi(x) - \Phi(x)\mathbf{a}_{in}(p')\,|\alpha_{in}\rangle \qquad \text{(II)} \end{aligned}$$

and again

$$\begin{aligned}(\text{II}) = &\lim_{y^0\to\infty}\frac{-i}{\sqrt{Z}}\int d^3y\,\langle\gamma_{out}|\,T(\Phi(y)\Phi(x))\,|\alpha_{in}\rangle\overleftrightarrow{\partial_0}e^{ip'y} - \\ &\lim_{y^0\to-\infty}\frac{-i}{\sqrt{Z}}\int d^3y\,\langle\gamma_{out}|\,T(\Phi(y)\Phi(x))\,|\alpha_{in}\rangle\overleftrightarrow{\partial_0}e^{ip'y} \\ =&\frac{i}{\sqrt{Z}}\int d^4y\,\langle\gamma_{out}|\,T(\Phi(y)\Phi(x))\,|\alpha_{in}\rangle\,(\overleftarrow{\partial}_y^2+m^2)e^{ip'y}\end{aligned}$$

and we ignore term (I) again, since it does not pertain to the interaction.

Applying this reduction algorithm until the in- and out-states have been reduced to the vacuum, and every time ignoring disconnected parts, gives the following general result:

$$\begin{aligned}&\langle p_1\dots p_n\text{ out}|q_1\dots q_m\text{ in}\rangle = \left(\frac{i}{\sqrt{Z}}\right)^{n+m}\int\prod_{i=1}^{m}d^4x_i\int\prod_{j=1}^{m}d^4y_j\times \qquad (5.19)\\ &e^{-iq_ix_i}(\vec{\partial}_{x_i}^2+m^2)\,\langle 0|\,T(\Phi(y_1),\dots,\Phi(y_n),\Phi(x_1),\dots,\Phi(x_m))\,|0\rangle\,(\overleftarrow{\partial}_{y_j}^2+m^2)e^{ip_jy_j}\end{aligned}$$

which, although rather long, does look nicely symmetric.

5.5.1 Generating functional

In elementary particle physics, one is mostly interested in S-matrix elements, to calculate scattering amplitudes and eventually cross sections. These can be related to time-ordered vacuum expectation values, as we have seen:

$$\langle 0|\,T(\Phi(x_1),\dots,\Phi(x_n)\,|0\rangle$$

Vacuum expectation values can be obtained by differentiating the *generating functional* $Z(j)$:

$$\boxed{Z(j) = \langle 0|\,T\left(exp(i\int d^4x\Phi(x)j(x)\right)|0\rangle} \qquad (5.20)$$

since

$$\left(\frac{1}{i}\right)^n\frac{\delta Z}{\delta j(x_1)\dots\delta j(x_n)}\bigg|_{j=0} = \langle 0|\,T[\Phi(x_1),\dots,\Phi(x_n)]\,|0\rangle$$

Remarks

- The S-matrix, which really is a unitary operator, can be expressed in terms of $Z(j)$ as well:

$$S =: exp\left(\int d^4y\left\{\Phi_{in}(y)\left[(\partial_y^2+m^2)\frac{1}{\sqrt{Z}}\frac{\delta}{\delta j(y)}\right]\right\}\right) : Z(j)\Big|_{j=0}$$

(again with normal ordering: annihilation operators to the right of creation operators). **Exercise**: show this.

- The T-product vacuum expectation values are the objectives of axiomatic QFT, which postulates spectral conditions, Lorentz/Poincaré invariance and microcausality, and seeks to build up its theories from these postulates.

- In perturbation theory, the Fourier transform of $\langle 0| T(\Phi(x_1), \ldots, \Phi(x_n)) |0\rangle$ produces poles in the outer momenta, which have to be truncated by the Klein-Gordon differential operators.

Chapter 6

Invariant Perturbation Theory

6.1 Dyson expansion, Gell-Mann-Low formula

To obtain the Gell-Mann-Low formula, one postulates a unitary operator $\mathbf{U}(t)$ which transforms Φ_{in} and Π_{in} into Φ and Π:

$$\boxed{\Phi(\vec{x},t) = \mathbf{U}^{-1}(t)\Phi_{\text{in}}(\vec{x},t)\mathbf{U}(t)} \tag{6.1}$$

$$\boxed{\Pi(\vec{x},t) = \mathbf{U}^{-1}(t)\Pi_{\text{in}}(\vec{x},t)\mathbf{U}(t)} \tag{6.2}$$

This would allow for canonical commutation relations for Φ and Φ_{in}. However, the representation of the commutators is unique up to unitary equivalence *only* in QM, where the number of degrees of freedom is finite.

The Hamiltonian $\mathbf{H}$ can be split up into two terms, one for the interaction and one for the free part:

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_{\text{int}}$$

These are all operator-valued functions of field operators; for example, in Φ^4-theory, $\mathbf{H}_{\text{int}} = \frac{\lambda}{4!}\int d^3x \Phi^4(\vec{x},t)$. The Heisenberg equations give the time developments of Φ_{in} and Φ as the equal-time commutators with the relevant Hamiltonians:

$$\frac{\partial \Phi_{\text{in}}(x)}{\partial t} = i[\mathbf{H}_0(\Phi_{\text{in}},\Pi_{\text{in}}),\Phi_{\text{in}}]_{\text{ET}} \tag{6.3}$$

$$\frac{\partial \Phi(x)}{\partial t} = i[\mathbf{H}(\Phi,\Pi),\Phi]_{\text{ET}} \tag{6.4}$$

Plugging eq. (6.1) allows us to rewrite the left-hand side of eq. (6.4) as

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \mathbf{U}^{-1}}{\partial t}\Phi_{\text{in}}\mathbf{U}(t) + \mathbf{U}^{-1}(t)\frac{\partial \Phi_{\text{in}}}{\partial t}\mathbf{U}(t) + \mathbf{U}^{-1}\Phi_{\text{in}}\frac{\partial \mathbf{U}}{\partial t} \tag{6.5}$$

We know the middle term of eq. (6.5) from eq. (6.3). Sandwiching eq. (6.5) between $\mathbf{U}(t)$ and $\mathbf{U}^{-1}(t)$ gives:

$$\mathbf{U}(t)\frac{\partial\Phi}{\partial t}\mathbf{U}^{-1}(t) = \mathbf{U}(t)\frac{\partial\mathbf{U}^{-1}(t)}{\partial t}\Phi_{\text{in}} + i[\mathbf{H}_0(\Phi_{\text{in}},\Pi_{\text{in}}),\Phi_{\text{in}}] + \Phi_{\text{in}}\frac{\partial\mathbf{U}(t)}{\partial t}\mathbf{U}^{-1}(t)$$

Since $\mathbf{H}$ is a polynomial in Φ and Π, for which we have eqs. (6.1) and (6.2), respectively, we know

$$\mathbf{U}(t)\frac{\partial\Phi}{\partial t}\mathbf{U}^{-1}(t) = i\mathbf{U}(t)[\mathbf{H}(\Phi,\Pi),\Phi]\mathbf{U}^{-1} = i[\mathbf{H}(\Phi_{\text{in}},\Pi_{\text{in}}),\Phi_{\text{in}}]$$

and so we can conclude that

$$\mathbf{U}(t)\frac{\partial\Phi}{\partial t}\mathbf{U}^{-1}(t) = i[\mathbf{H}(\Phi_{\text{in}},\Pi_{\text{in}}),\Phi_{\text{in}}]$$

Since $\mathbf{U}$ is unitary, we know that

$$\mathbf{U}\mathbf{U}^{-1} = 1 \quad \text{and hence} \quad \mathbf{U}\partial_t\mathbf{U}^{-1} + (\partial_t\mathbf{U})\mathbf{U}^{-1} = 0$$

Using $\mathbf{H} - \mathbf{H}_0 = \mathbf{H}_{\text{int}}$ and the above gives

$$i[\mathbf{H}_{\text{int}}(\Phi_{\text{in}},\Pi_{\text{in}}),\Phi_{\text{in}}] = -[(\partial_t\mathbf{U}(t))\mathbf{U}^{-1}(t),\Phi_{\text{in}}] \tag{6.6}$$

where we could in many cases also write $\mathbf{H}_{\text{int}}(\Phi_{\text{in}})$, since $\mathbf{H}_{\text{int}}$ usually does not depend on Π_{in} (e.g. $\mathbf{H}_{\text{int}} = \frac{\lambda}{4!}\int d^3x\Phi^4(\vec{x},t)$). So, up to constants that are of no interest to us, we have

$$(\partial_t\mathbf{U}(t))\mathbf{U}^{-1}(t) = -i\mathbf{H}_{\text{int}}(\Phi_{\text{in}})$$

Defining

$$\mathbf{U}(t,t') := \mathbf{U}(t)\mathbf{U}^{-1}(t') \tag{6.7}$$

we have

$$\boxed{\frac{\partial\mathbf{U}(t,t')}{\partial t} = -i\mathbf{H}_{\text{in}}(\Phi_{\text{in}}(t))\mathbf{U}(t,t')} \tag{6.8}$$

which is solved by

$$\boxed{\mathbf{U}(t,t') = \mathrm{T}\left(\exp\left[-i\int_{t'}^{t}dt''\mathbf{H}_{\text{int}}(\Phi_{\text{in}}(t''))\right]\right)} \tag{6.9}$$

where the time-ordering applies to the terms in the exponential series.

Exercise: show that this is a solution. Hint: transfer into an integral equation:

$$\mathbf{U}(t,t') = 1 - i\int_{t'}^{t}dt''\mathbf{H}_{\text{int}}(\Phi_{\text{in}}(t''))\mathbf{U}(t'',t')$$

Solve this integral equation by iteration and doing the nested integrations over a generalized "box", instead of a "tetrahedron", using the time ordering

prescription (of any two fields, the one at the prior time is to the right of the one to the left). Alternatively, check that $\mathbf{U}(t_1,t_2)\mathbf{U}(t_2,t_3)=\mathbf{U}(t_1,t_3)$; then, with $t_1-t_2=\Delta t$, the differential equation (eq. (6.8)) is fulfilled.

Note: we avoid, unlike in the case presented by Itzykson & Zuber, $\mathbf{U}(t)=\mathrm{T}(\int_{-\infty}^{t}\dots)$, which, with $\mathbf{U}(-\infty)=t$ would give $\Phi(-\infty)=\Phi_{\mathrm{in}}$.

Now, assume $x_1^0\geq x_2^0\geq\dots$ (if this is not the case, rename your x so that it is the case). Then,

$$\begin{aligned}\langle 0|\,\mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle = \langle 0|\,\mathbf{U}^{-1}(t)\mathbf{U}(t,t_1)\Phi_{\mathrm{in}}(x_1)\mathbf{U}(t_1,t_2)\cdots\times\\ \dots\Phi_{\mathrm{in}}(x_n)\mathbf{U}(t_n,t)\mathbf{U}(-t)\,|0\rangle\end{aligned}$$

Now taking $t\to\infty$ gives

$$\lim_{t\to\infty}\langle 0|\,\mathbf{U}^{-1}(t)\mathrm{T}\left(\Phi_{\mathrm{in}}(x_1)\dots\Phi_{\mathrm{in}}(x_n)\exp\left\{-i\int_{-t}^{t}dt'\mathbf{H}_{\mathrm{int}}(t')\right\}\right)\mathbf{U}(-t)\,|0\rangle$$

where again the exponential is to be split up into time-ordered parts. Now consider

$$\lim_{t\to\infty}\mathbf{U}(-t)\,|0\rangle\stackrel{!}{=}\lambda_-\,|0\rangle\quad\text{and}\quad\lim_{t\to\infty}\mathbf{U}(t)\,|0\rangle\stackrel{!}{=}\lambda_+\,|0\rangle$$

where the λ's are phase factors resulting from $\mathbf{U}$'s unitarity. The proof of these two limits is as follows (compare Björken & Drell, vol. II, p. 186): assume

$$\lim_{t\to\infty}\langle\beta|\,\mathbf{U}(-t)\,|0\rangle\neq 0$$

where $\langle\beta|\neq\langle 0|$. Then take a particle with momentum p out of $\langle\beta|$:

$$\begin{aligned}\langle p\,\alpha_{\mathrm{in}}|\,\mathbf{U}(-t)\,|0\rangle=\langle\alpha_{\mathrm{in}}|\,\mathbf{a}_{\mathrm{in}}(p)\mathbf{U}(-t)\,|0\rangle=\\ -i\int d^3x e^{-ip_0t'-i\vec{p}\cdot\vec{x}}\frac{\overleftrightarrow{\partial}}{\partial x'^0}\langle\alpha_{\mathrm{in}}|\,\Phi_{\mathrm{in}}(-t',\vec{x})\mathbf{U}(-t)\,|0\rangle\end{aligned}$$

and note that $\Phi_{\mathrm{in}}(-t',\vec{x})\mathbf{U}(-t)=\mathbf{U}(-t')\Phi(-t',\vec{x})\mathbf{U}^{-1}(-t')\mathbf{U}(-t)$. Now let $t=t'\to\infty$; this gives

$$\begin{aligned}=&\sqrt{Z}\,\langle\alpha_{\mathrm{in}}|\,\mathbf{U}(-t)\overbrace{\mathbf{a}_{\mathrm{in}}(p)\,|0\rangle}^{=0}+i\int d^3x e^{-ip_0t-i\vec{p}\cdot\vec{x}}\times\\ &\langle\alpha_{\mathrm{in}}|\,(\partial_t\mathbf{U}(-t))\Phi(-t,\vec{x})+\mathbf{U}(-t)\Phi(-t,\vec{x})(\partial_t\mathbf{U}^{-1}(-t))\mathbf{U}(-t)\,|0\rangle\end{aligned}$$

which gives zero. To see this (exercise), use the unitary equivalence and the equation for $\partial_t\mathbf{U}$ derived above. Now, since we had assumed it to be nonzero, this is a contradiction.

Now, we obtain $\lambda_-\lambda_+^*$ as follows:

$$1 = \langle 0|0\rangle = \lim_{t\to\infty} \langle 0|\, \mathbf{U}^{-1}(t)\, \underbrace{\mathbf{U}(t)\mathbf{U}^{-1}(-t)}_{\mathbf{U}(t,-t)}\, \mathbf{U}(-t)\, |0\rangle =$$

$$\lambda_-\lambda_+^*\, \langle 0|\, \mathbf{U}(\infty,-\infty)\, |0\rangle =$$

$$\lambda_-\lambda_+^*\, \langle 0|\, \mathrm{T}\exp\left[-i\int_{-\infty}^{\infty} dt \mathbf{H}_{\mathrm{int}}(\Phi_{\mathrm{in}}(t))\right] |0\rangle$$

which leads us to our final result: the Gell-Mann-Low formula:

$$\langle 0|\, \mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\, |0\rangle = \frac{\langle 0|\, \mathrm{T}\left(\Phi_{\mathrm{in}}(x_1)\dots\Phi_{\mathrm{in}}(x_n)\exp\left[-i\int_{-\infty}^{\infty} dt \mathbf{H}_{\mathrm{int}}(\Phi_{\mathrm{in}}(t))\right]\right) |0\rangle}{\langle 0|\exp\left[-i\int_{-\infty}^{\infty} dt \mathbf{H}_{\mathrm{int}}(\Phi_{\mathrm{in}}(t))\right] |0\rangle} \tag{6.10}$$

This will be our basis when doing perturbation theory. Note that we have one vacuum $|0\rangle$, and that everything is written in terms of free Φ_{in}-fields, obtained via the Yang-Feldman equations, acting in Fock space.

6.2 Interaction picture

For completeness' sake, we will present the interaction picture (also known as Dirac picture), since many textbooks use it to derive the Gell-Mann-Low formula. It also offers more insight in the differences between the different pictures.

6.2.1 Transformation

In the interaction picture, the time dependence is divided over the operators and states, unlike in the Heisenberg and Schrödinger pictures. The time developments of those parts of operators that are not explicitly time-dependent are given by $\mathbf{H}_0$; those of states by $\mathbf{H}_{\mathrm{int}}$. In the Schrödinger picture,

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_{\mathrm{int}}$$

At $t = 0$, all three pictures (Heisenberg, Schrödinger and Dirac) are assumed to coincide. From there, we obtain field operators in the interaction picture as follows:

$$\Phi_0(\vec{x}, t) = e^{i\mathbf{H}_0 t}\Phi(\vec{x}, 0)e^{-i\mathbf{H}_0 t} \tag{6.11}$$

The Heisenberg analogue is

$$\Phi(\vec{x}, t) = e^{i\mathbf{H}t}\Phi(\vec{x}, 0)e^{-i\mathbf{H}t} \tag{6.12}$$

Note that $\mathbf{H}$ is time independent in *all* pictures.

The transformation from the Dirac to the Heisenberg picture,

$$\Phi(\vec{x},t) = \underbrace{e^{i\mathbf{H}t}e^{-i\mathbf{H}_0 t}}_{\widetilde{\mathbf{U}}^\dagger(t)}\,\Phi_0(\vec{x},t)\,\underbrace{e^{i\mathbf{H}_0 t}e^{-i\mathbf{H}t}}_{\widetilde{\mathbf{U}}(t)} \tag{6.13}$$

defines a unitary operator $\widetilde{\mathbf{U}}$ with the following property:

$$\frac{d\widetilde{\mathbf{U}}(t)}{dt} = -ie^{i\mathbf{H}_0 t}(\mathbf{H}-\mathbf{H}_0)e^{i\mathbf{H}t} = -ie^{i\mathbf{H}_0 t}\mathbf{H}_{\text{int}}e^{i\mathbf{H}t} \tag{6.14}$$

Defining $\mathbf{H}'_{\text{int}} := e^{i\mathbf{H}_0 t}\mathbf{H}_{\text{int}}e^{-i\mathbf{H}_0 t}$, we can rewrite this as

$$\frac{d\widetilde{\mathbf{U}}(t)}{dt} = -ie^{i\mathbf{H}_0 t}\mathbf{H}_{\text{int}}e^{-i\mathbf{H}_0 t}e^{i\mathbf{H}_0 t}e^{i\mathbf{H}t} = -i\mathbf{H}'_{\text{int}}e^{i\mathbf{H}_0 t}e^{-i\mathbf{H}t} = -i\mathbf{H}'_{\text{int}}\widetilde{\mathbf{U}}(t)$$

This gives for $\widetilde{\mathbf{U}}(t)$, like for $\mathbf{U}(t,t')$ before, the following iterative formula:

$$\widetilde{\mathbf{U}}(t) = 1 - i\int_0^t dt'\mathbf{H}'_{\text{int}}(t')\widetilde{\mathbf{U}}(t') \tag{6.15}$$

From this, one can again obtain the Gell-Mann-Low formula, this time with expectation values in the vacuum $\left|\tilde{0}\right\rangle$ of the interaction picture, for which $\mathbf{H}_0\left|\tilde{0}\right\rangle = 0$. Note that $\mathbf{H}'_{\text{int}}$ is a polynomial of field operators of the interaction picture.

$\mathbf{H}_0$ now is the free Hamiltonian with the mass term $\frac{1}{2}m_0^2\Phi_0^2$ from the original (also called "bare") Lagrangian. This is generally different from the $\mathbf{H}_0$ we had before, which gives the time dependence of $\Phi_{\text{in}}(x)$ and has a physical mass m^2. The notation $\mathbf{H}_0^{(m)}$ is also used.

Starting again from the T-product vacuum expectation value (vev)

$$\langle 0|\,\mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle$$

and transforming to Φ_0 we now want to know the limit (in the weak sense) of $\widetilde{\mathbf{U}}^\dagger(T)\left|\tilde{0}\right\rangle$ as T goes to $\pm\infty$. Shifting the energy such that both $\mathbf{H}_0\left|\tilde{0}\right\rangle = 0$ and $\mathbf{H}\left|0\right\rangle = 0$, one has

$$\langle\Psi|\,e^{i\mathbf{H}T}e^{-i\mathbf{H}_0 T}\left|\tilde{0}\right\rangle = \langle\Psi|\,e^{i\mathbf{H}T}\left|\tilde{0}\right\rangle = \\ \langle\Psi|0\rangle\left\langle 0|\tilde{0}\right\rangle + \int dE\,\langle\Psi|E\rangle\left\langle E|\tilde{0}\right\rangle e^{iET}$$

where $\int dE\,\langle\Psi|E\rangle\left\langle E|\tilde{0}\right\rangle e^{iET} \to 0$ for $T\to\pm\infty$ by the Riemann-Lebesgue lemma. From this, we read off

$$\widetilde{\mathbf{U}}^\dagger(T)\left|\tilde{0}\right\rangle \rightharpoonup |0\rangle\left\langle 0|\tilde{0}\right\rangle \quad \text{as} \quad T\to\pm\infty \tag{6.16}$$

which is the Gell-Mann-Low theorem.

Using

$$\langle 0|0\rangle = 1 = \lim_{T\to\infty} \frac{\langle\tilde{0}|\,\widetilde{\mathbf{U}}(T)\widetilde{\mathbf{U}}^\dagger(-T)\,|\tilde{0}\rangle}{|\,\langle 0|\tilde{0}\rangle\,|^2}$$

we can solve for $|\,\langle 0|\tilde{0}\rangle\,|^2$ and obtain

$$\langle 0|\,\mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle = \tag{6.17}$$
$$\lim_{T\to\infty} \frac{\langle\tilde{0}|\,\mathrm{T}\left(\Phi_0(x_1)\dots\Phi_0(x_n)\exp\left[-i\int_{-T}^{T}\mathbf{H}'_{\mathrm{int}}(t)dt\right]\right)|\tilde{0}\rangle}{\langle\tilde{0}|\,\mathrm{T}\left(\exp\left[-i\int_{-T}^{T}\mathbf{H}'_{\mathrm{int}}(t)dt\right]\right)|\tilde{0}\rangle}$$

if we substitute $|0\rangle$ according to eq. (6.16) and plug in eq. (6.13) for Φ.

6.2.2 Scattering theory

If one wants to start the discussion of the interaction picture from the beginning, one has to introduce scattering states, like in QM. States in the interaction picture have a time development:

$$i\frac{d}{dt}\,|\psi\rangle_I = \mathbf{V}_I(t)\,|\psi\rangle_I \tag{6.18}$$

Note that $\mathbf{V}_I$ was called $\mathbf{H}'_{\mathrm{int}}$ in the previous section, so

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{V}_{\mathrm{Sch}} \tag{6.19}$$

and

$$\mathbf{V}_I = e^{i\mathbf{H}_0 t}\mathbf{V}_{\mathrm{Sch}}e^{-i\mathbf{H}_0 t} \tag{6.20}$$

For large positive or negative values of t the interaction is switched off. $|\psi(\pm\infty)\rangle$ are time independent and correspond to states in the Schrödinger picture changing in time with $\mathbf{H}_0$.

Naïvely, $\mathbf{H}$ and $\mathbf{H}_0$ are assumed to have the same spectrum and to act in the same Hilbert space. However, this has to be checked carefully; for example, the "bare" masses in $\mathbf{H}_0$ are not the physical masses appearing in the full $\mathbf{H}$ unless one shifts terms in the Hamiltonian $\mathbf{H}$ from $\mathbf{V}$ to $\mathbf{H}_0$ (adding and subtracting mass terms $\frac{1}{2}(m_0^2 - m^2)\Phi^2$).

Consider stationary states

$$\mathbf{H}_0\,|\phi_\alpha\rangle = E_\alpha\,|\phi_\alpha\rangle$$
$$\mathbf{H}\,|\psi_\alpha^\pm\rangle = E_\alpha\,|\psi_\alpha^\pm\rangle$$

where the scattering states $|\psi_\alpha^\pm\rangle$ serve to construct wave packets which approach $|\phi_\alpha\rangle$ for $t \to \mp\infty$ (see figure 6.1); i.e.

$$\boxed{\int d\alpha\, e^{-iE_\alpha t}g(\alpha)\,|\psi_\alpha^\pm\rangle \to \int d\alpha\, e^{-iE_\alpha t}g(\alpha)\,|\phi_\alpha\rangle} \tag{6.21}$$

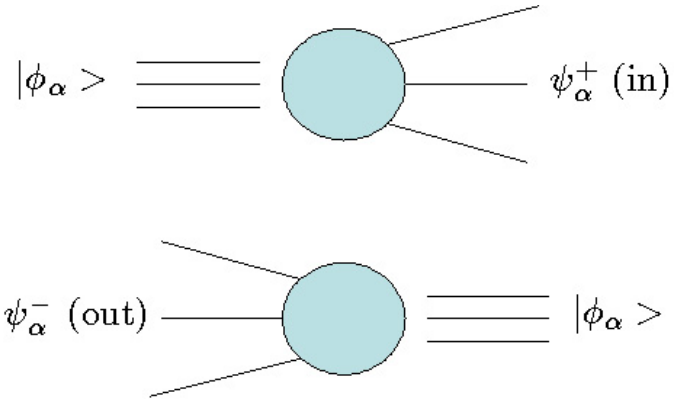


Figure 6.1: $|\phi_\alpha\rangle$ as $t \to \pm\infty$

Formally, this means

$$\left|\psi_\alpha^\pm\right\rangle = \Omega(\mp\infty)\,|\phi_\alpha\rangle \quad \text{or}$$
$$|\phi_\alpha\rangle = \Omega^\dagger(\mp\infty)\left|\psi_\alpha^\pm\right\rangle$$

with Hermitian $\Omega(t) = e^{i\mathbf{H}t}e^{-i\mathbf{H}_0 t}$, also called the Møller operator. Now, it is easy to check that

$$i\frac{d}{dt}\Omega^\dagger(t) = \mathbf{V}_I\Omega^\dagger(t)$$

which is solved by

$$\Omega^\dagger(t) = \mathrm{Texp}\left(-i\int_0^t dt\mathbf{V}_I(t')\right)$$

(where $\Omega^\dagger(0) = 1$). Using this Ω, a solution to eq. (6.18) can be written

$$|\psi(\infty)\rangle_I = \Omega^\dagger(\infty)\Omega(-\infty)\underbrace{|\psi(-\infty)\rangle_I}_{=|\phi_\alpha\rangle}$$

with the **S**-*matrix*:

$$\boxed{\mathbf{S} = \Omega^\dagger(\infty)\Omega(-\infty) = \mathrm{Texp}\left(-i\int_{-\infty}^{\infty} dt'\mathbf{V}_I(t')\right)} \tag{6.22}$$

S-matrix elements are given by

$$\left\langle\psi_\beta^-|\psi_\alpha^+\right\rangle = \langle\phi_\beta|\,\Omega^\dagger(\infty)\Omega(-\infty)\,|\phi_\alpha\rangle = \langle\phi_\beta|\,\mathbf{S}\,|\phi_\alpha\rangle \tag{6.23}$$

Evaluating the **S**-matrix operator with the Wick theorem will also lead to invariant pertubation theory and the Feynman rules.

Note

If one wants to stay close to QM, one considers the *Lippmann - Schwinger* equations for $|\psi_\alpha^\pm\rangle$:

$$\left|\psi_\alpha^\pm\right\rangle = |\phi_\alpha\rangle + (E_\alpha - \mathbf{H}_0 \pm i\epsilon)^{-1}\mathbf{V}\left|\psi_\alpha^\pm\right\rangle \tag{6.24}$$

which on the face of it looks like a simple identity, but contains the asymptotes from eq. (6.21).

The evaluation of the **S**-matrix elements could also be done in *time ordered perturbation theory*, with "on-shell" intermediate states. To get the right $i\epsilon$ prescriptions, one goes to the Lippmann-Schwinger equations and introduces

$$\langle\beta|\,\mathbf{V}\left|\psi_\alpha^+\right\rangle = \mathbf{T}^+_{\beta\alpha} \qquad \text{fulfilling}$$
$$\mathbf{T}^+_{\beta\alpha} = \mathbf{V}_{\beta\alpha} + \int d\gamma \mathbf{V}_{\beta\gamma}\mathbf{T}^+_{\gamma\alpha}(E_\alpha - E_\gamma + i\epsilon)$$

Then, one has

$$\mathbf{S}_{\alpha\beta} = \delta(\beta - \alpha) - 2\pi i\delta(E_\alpha - E_\beta)\mathbf{T}^+_{\beta\alpha}$$

The equation for $\mathbf{T}^+_{\beta\alpha}$ can be solved by a Neumann series. This results in a set of rules equivalent to the Feynman rules; later on, we will obtain the denominators by integrating k_0 in the Feynman graphs.

6.2.3 Background field

If one has just an outer field j (also called background field), one has to distinguish $|0\rangle_{\text{in}}$ and $|0\rangle_{\text{out}}$, but apart from this, the reduction formula works like the LSZ formalism. One then has to consider

$$\langle 0|_{\text{out}}\,\mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle_{\text{in}} \tag{6.25}$$
$$\text{with} \qquad \Phi(x) = \mathbf{U}^{-1}(t)\Phi_{\text{in}}(x)\mathbf{U}(t)$$
$$\text{and} \qquad \mathbf{U}(t) = \mathrm{Texp}\left(-i\int_{-\infty}^{t} dt'\mathbf{H}_{\text{int}}(t')\right)$$

where $\mathbf{H}_{\text{int}} = \int d^3x\Phi(x)j(x)$. Now, $\langle 0|_{\text{out}}$ and $\langle 0|_{\text{in}}$ are related by

$$\langle 0|_{\text{out}} = \langle 0|_{\text{in}}\,\mathbf{S}$$

with

$$\mathbf{S} = \mathrm{Texp}\left(-i\int_{-\infty}^{\infty} dt'\mathbf{H}_{\text{int}}(t')\right) \tag{6.26}$$

Then, (6.25) reads:

$$\lim_{T\to\infty} \langle 0|_{\text{in}} \,\text{Texp}\left(-i\int_{-\infty}^{\infty} dt' \mathbf{H}_{int}(t')\right) \mathbf{U}^{-1}(T)\mathbf{U}(T)\mathbf{U}^{-1}(t)\times$$
$$\Phi_{\text{in}}(x_1)\mathbf{U}(t_1,t_2)\ldots\Phi_{in}(x_n)\mathbf{U}(t_n)\mathbf{U}^{-1}(T)\mathbf{U}(T)\,|0\rangle =$$
$$\langle 0|_{\text{in}} \,\text{T}\left(\Phi_{\text{in}}(x_1)\ldots\Phi_{\text{in}}(x_n)\text{exp}\left[-i\int_{-\infty}^{\infty} dt' \mathbf{H}_{\text{int}}(t')\right]\right)|0\rangle_{\text{in}} \qquad (6.27)$$

This allows us to discuss the production of n particles out of the vacuum by an outer source (like a current) $j(x)$ switched on at some point in time, and switched off at another. The calculation of the **S**-matrix transition amplitudes and of transition probabilities leads to a Poisson distribution. For more on this topic, see Itzykson and Zuber, p. 176.

Since this is a tree-level calculation, without loops, $Z \neq 1$ is not necessary here, so we can caculate the amplitude directly:

$$\langle (p_1\ldots p_n)_{\text{out}}|0\rangle_{\text{in}} = {}_{\text{in}}\langle p_1\ldots p_n|\,\mathbf{S}\,|0\rangle_{\text{in}}$$

with

$$\mathbf{S} = \text{Texp}\left(-i\int_{-\infty}^{\infty} dt' \int d^3x(-j(\vec{x},t')\Phi_{\text{in}}(\vec{x},t'))\right)$$

Splitting into time intervals and using $e^A e^B = e^{A+B+[A,B]/2}$, this becomes

$$\mathbf{S} = \exp\left(-i\int d^4x \Phi_{\text{in}}(x)\; j(x)\right)\exp\left(-\frac{1}{2}\int\int d^4x d^4y \theta(x^0-y^0)[\Phi_{\text{in}}(x),\Phi_{\text{in}}(y)]j(x)j(y)\right)$$

- Only the $\mathbf{a}^\dagger$-part of $\Phi_{in}(x)$ contributes to the first part; the second part is a common factor

  $$\exp\left(-\frac{1}{2}\int\int d^4x d^4y D_R(x-y)j(x)j(y)\right) = \exp\left(-\frac{1}{2}\int \widetilde{dk}|\tilde{j}(k)|^2\right)$$

  with $|\tilde{j}(k)|^2 = \tilde{j}(k)\tilde{j}^*(k) = \tilde{j}(k)\tilde{j}(-k)$.

- **S** produces a coherent state out of $|0\rangle_{in}$.

- The probabilities for producing n particles (see Itzykson & Zuber, p. 169)

  $$p_n = \frac{1}{n!}\left(\int \widetilde{dq}|j(q)|^2\right)^n \exp\left(-\int \widetilde{dq}|j(q)|^2\right)$$

  form a Poisson distribution.

6.3 Haag's theorem

Assume

(i) translational invariance of the physical vacuum (i.e. the one from the Heisenberg picture)

(ii) "vacuum polarization", i.e. $\left|\tilde{0}\right\rangle$ is not an eigenstate of $\mathbf{H}$

Then, one finds that $\widetilde{\mathbf{U}}(t)$ does not exist. In other words, the interaction picture does not exist in QFT.

This is a typical problem with the limit $V \to \infty$. It also appears in the thermodynamical limit of statistical mechanics (see also R. Haag, *Local Quantum Physics*, Springer monographs in physics). The essential point is that $\mathbf{H}$ does not exist in the Fock space $\mathfrak{H}_F$ of the $\mathbf{H}_0$-states. $\mathbf{H}$ and $\mathbf{H}_0$ act in non-equivalent Fock spaces; they exist as different irreducible representations of the commutation relations in the case of an infinite number of degrees of freedom.

Sketch of a proof (see Hegerfeld, Göttingen): define

$$\mathbf{H}\left|\tilde{0}\right\rangle = \lim_{V\to\infty} \int d^3x (\mathfrak{H}_0(x) + \mathfrak{H}_{int}(x)) \left|\tilde{0}\right\rangle =: |\psi\rangle$$

By assumtion (ii), $|\psi\rangle \neq \alpha \left|\tilde{0}\right\rangle$. Now, from the Gell-Mann-Low theorem

$$|0\rangle = \lim_{t\to\pm\infty} \frac{\widetilde{\mathbf{U}}^\dagger(t)\left|\tilde{0}\right\rangle}{\langle 0|\,\widetilde{\mathbf{U}}^\dagger(t)\left|\tilde{0}\right\rangle}$$

we see that $\left|\tilde{0}\right\rangle$ has to be translationally invariant, because $|0\rangle$ is (assumption (i)). Hence

$$\left\langle\tilde{0}\right|\mathfrak{H}(x)\,|\psi\rangle = C \text{ (const)}$$

since both $\left\langle\tilde{0}\right|$ and $|\psi\rangle$ are translationally invariant. Therefore,

$$\langle\psi|\psi\rangle = \lim_{V\to\infty} \int d^3x \left\langle\tilde{0}\right|\mathfrak{H}(x)\,|\psi\rangle = \infty$$

for $C \neq 0$.

Alternatively, one might use that $\langle\psi|\psi\rangle$ is finite, so C must be zero, so $\langle\psi|\psi\rangle = 0$, which means that $|\phi\rangle = \mathbf{H}\left|\tilde{0}\right\rangle = 0$, which contradicts assumption (i).

Chapter 7

Feynman rules, cross section

7.1 Wick theorem

Last chapter, we ended with the Gell-Mann-Low formula, which contained time ordered products of $\Phi_{\text{in}}(x)$ ($\mathbf{H}_{\text{int}}$ is also an expression in Φ_{in}). We want to rewrite these in terms of c number valued "functions", which will later on turn out to be distributions, and *normal ordered products*, where the annihilation operators are placed to the right of the creation operators. The latter do not contribute to vacuum expectation values, but they are still important in time ordered perturbation theory, where they act on Φ_0-states.

In this chapter, we will omit the subscript "in" from Φ_{in}, and simply write Φ. Now, let us start by considering

$$\mathrm{T}(\Phi(x_1)\Phi(x_2))$$

Recall that $\Phi(x) = \Phi^-(x) + \Phi^+(x)$, where $\Phi^-(x)$ contains the generation operator, and $\Phi^+(x)$ the annihilation operator. Further, assume that $x_1^0 \geq x_2^0$. Then,

$$\begin{aligned}\mathrm{T}(\Phi(x_1)\Phi(x_2)) = {} & \Phi^-(x_1)\Phi^-(x_2) + \Phi^+(x_1)\Phi^+(x_2) + \\ & \Phi^-(x_1)\Phi^+(x_2) + \Phi^+(x_1)\Phi^-(x_2)\end{aligned}$$

In this expression, the first three terms on the right hand side are already written in their normal ordered form. The fourth term, $\Phi^+(x_1)\Phi^-(x_2)$, still has the annihilation operator to the left of the creation operator. Introducing the normal ordering, this term has to be removed:

$$\begin{aligned}&\mathrm{T}(\Phi(x_1)\Phi(x_2)) = \\ &:\Phi(x_1)\Phi(x_2): + \Phi^+(x_1)\Phi^-(x_2) - \Phi^-(x_2)\Phi^+(x_1) = \\ &:\Phi(x_1)\Phi(x_2): + D^+(x_1 - x_2)\theta(x_1^0 - x_2^0) \qquad (7.1)\end{aligned}$$

$D^+(x_1 - x_2) = \langle 0| \Phi(x_1)\Phi(x_2) |0\rangle$ gives the first part of $D_{\rm C}$. In the case where $x_2^0 \geq x_1^0$, i.e. where 1 and 2 are exchanged, gives

$$\begin{aligned} \mathrm{T}(\Phi(x_1)\Phi(x_2)) = &: \Phi(x_1)\Phi(x_2) : + D^+(x_2 - x_1)\theta(x_2^0 - x_1^0) = \\ &: \Phi(x_1)\Phi(x_2) : - D^-(x_1 - x_2)\theta(x_2^0 - x_1^0) \end{aligned} \tag{7.2}$$

which is the second part of $D_{\rm C}$. To see the connection to $D^{\pm}$, recall the definition of $\Phi(x)$:

$$\Phi(x) = \int \tilde{dk} \left(\mathbf{a}(k) e^{-ikx} + \mathbf{a}^\dagger e^{ikx} \right)$$

(where $\Phi^+ = \int \widetilde{dk} \mathbf{a}(k) e^{-ikx}$). Then, for $x_1^0 \geq x_2^0$, the commutator of $\Phi^+(x_1)$ and $\Phi^-(x_2)$ gives

$$\begin{aligned} [\Phi^+(x_1), \Phi^-(x_2)] = &\int \tilde{dk}_1 \tilde{dk}_2 e^{-ik_1x_1 + ik_2x_2} [\mathbf{a}(k_1), \mathbf{a}^\dagger(k_2)] = \\ &\int \tilde{dk}_1 e^{-ik_1(x_1 - x_2)} = D^+(x_1 - x_2) = \\ &\langle 0| \Phi(x_1)\Phi(x_2) |0\rangle \end{aligned} \tag{7.3}$$

This is a c number. Taking the vacuum expectation value of $\mathrm{T}(\Phi(x_1)\Phi(x_2))$, the normal ordered part drops out, and the second part has exactly the structure of the Feynman propagator we have seen before. Therefore, the vacuum expectation value $\langle 0| \mathrm{T}(\Phi(x_1)\Phi(x_2)) |0\rangle$ can be written as

$$\langle 0| \mathrm{T}(\Phi(x_1)\Phi(x_2)) |0\rangle = \tag{7.4}$$

$$D^+(x_1 - x_2)\theta(x_1^0 - x_2^0) - D^-(x_1 - x_2)\theta(x_1^0 - x_2^0) = \tag{7.5}$$

$$D_{\rm F}(x_1 - x_2) = iG_{\rm F}(x_1 - x_2) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2 - m^2 + i\epsilon} \tag{7.6}$$

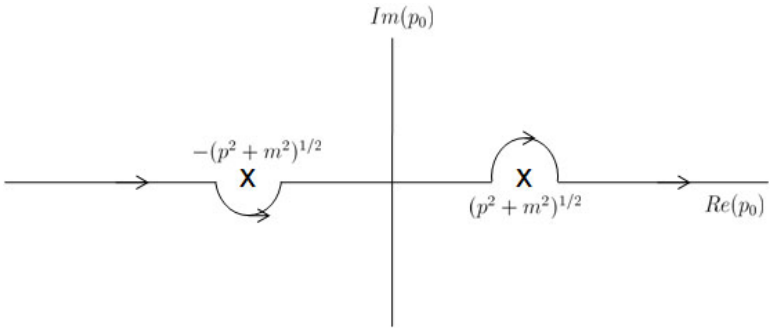


Figure 7.1: Integration path for $D_{\rm F}$

Now, Wick's theorem states that the general T-product can be reduced to normal ordered products and vacuum expectation values of T-products of *pairs* of Φ:

$$
\begin{aligned}
&\mathrm{T}(\Phi(x_1)\dots\Phi(x_n)) = \\
&:\Phi(x_1)\dots\Phi(x_n): + \\
&\langle 0|\,\mathrm{T}(\Phi(x_1)\Phi(x_2))\,|0\rangle : \Phi(x_3)\dots\Phi(x_n): + \cdots + \\
&\langle 0|\,\mathrm{T}(\Phi(x_1)\Phi(x_2))\,|0\rangle\,\langle 0|\,\mathrm{T}(\Phi(x_3)\Phi(x_n))\,|0\rangle : \Phi(x_5)\dots\Phi(x_n): + \cdots + \\
&\dots \quad \text{going over all possible contractions}
\end{aligned}
\tag{7.7}
$$

Here, $\langle 0|\,\mathrm{T}(\Phi(x_1)\Phi(x_n))\,|0\rangle$ is called the *contraction* of $\Phi(x_1)$ and $\Phi(x_2)$, which is often denoted by $\overline{\Phi(x_1)\Phi(x_2)}$.

The theorem is proven by induction: first, we see that it works for $n = 2$. Then, we prove that if the theorem holds for n, it also holds for $n + 1$. To begin the proof, let us note that without limitation of generality, we can assume that x^0_{n+1} is smaller than all other x^0, so

$$\mathrm{T}(\Phi(x_1)\dots\Phi(x_n)\Phi(x_{n+1})) = \mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\Phi(x_{n+1})$$

Now, by assumption, $\mathrm{T}(\Phi(x_1)\dots\Phi(x_n))$ can be written in terms of normal ordered products. Rewriting $\Phi(x_{n+1})$ as $\Phi^+(x_{n+1}) + \Phi^-(x_{n+1})$, we see that the term with Φ^+ is already normal ordered. The other term still has to be ordered, so $\Phi^-(x_{n+1})$ has to be commuted with all the Φ^+.

$$
\begin{aligned}
[\Phi^+(x_k), \Phi^-(x_{n+1})] = c\text{-number} = \langle 0|\,[\Phi^+(x_k), \Phi^-(x_{n+1})]\,|0\rangle = \\
\langle 0|\,\Phi^+(x_k)\Phi^-(x_{n+1})\,|0\rangle = \langle 0|\,\Phi(x_k)\Phi(x_{n+1})\,|0\rangle = \\
\langle 0|\,\mathrm{T}(\Phi(x_k)\Phi(x_{n+1}))\,|0\rangle
\end{aligned}
$$

Commuting with all the Φ^+ gives all possible contractions, as Wick's theorem says.

$\mathfrak{H}_{\mathrm{int}}$ is often assumed to be already normal ordered, e.g. $:\Phi^4(x):$. Otherwise, one would get "self-contractions", as we will see later when treating the path integral formulation.

7.1.1 Φ^4-theory

Evaluating interactions in Φ^4-theory by Wick's theorem highlights the central problem of local QFT: it involves polynomials in G_{F}. If G_{F} is a distribution (like a delta function), taking it to any power other than one is a

very tricky affair. Consider $\mathrm{T}(:\Phi^4(x)::\Phi^4(y):)$:

$$\begin{aligned}\mathrm{T}(:\Phi^4(x)::\Phi^4(y):) = &:\Phi^4(x)\Phi^4(y):+\\ &1!\begin{pmatrix}4\\1\end{pmatrix}^2\overline{\Phi(x)\Phi(y)}:\Phi^3(x)\Phi^3(y):+\\ &2!\begin{pmatrix}4\\2\end{pmatrix}^2\left(\overline{\Phi(x)\Phi(y)}\right)^2:\Phi^2(x)\Phi^2(y):+\\ &3!\begin{pmatrix}4\\3\end{pmatrix}^2\left(\overline{\Phi(x)\Phi(y)}\right)^3:\Phi(x)\Phi(y):+\\ &4!\left(\overline{\Phi(x)\Phi(y)}\right)^4 \end{aligned} \tag{7.8}$$

and note that the last three terms have $\left(\overline{\Phi(x)\Phi(y)}\right)$, which was shown to be D_{F}, to some power higher than 1.

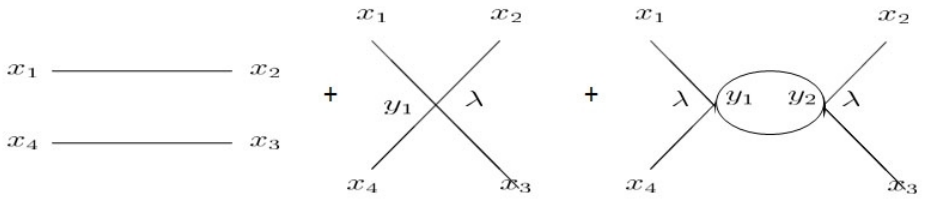
7.2 Feynman graphs

7.2.1 Feynman rules in x-space

We want to evaluate the Gell-Mann-Low formula for the T-product vacuum expectation value of interacting fields from last chapter:

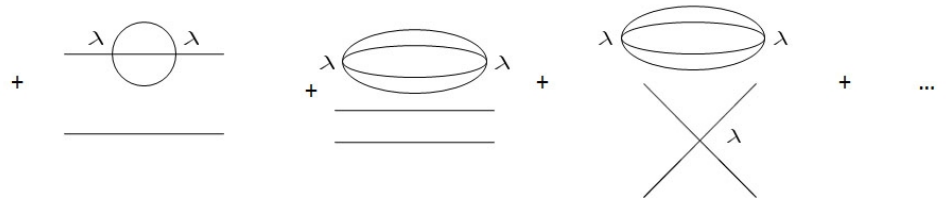
$$\frac{\langle 0|\,\mathrm{T}[\exp\left(-i\int d^4x\mathfrak{H}_{\mathrm{int}}(\Phi(x))\right)\Phi(x_1)\dots\Phi(x_n)]\,|0\rangle}{\langle 0|\,\mathrm{T}\left[\exp\left(-i\int d^4x\mathfrak{H}_{\mathrm{int}}(\Phi(x))\right)\right]|0\rangle}$$

For the sake of simplicity, let us restrict ourselves to real Φ^4-theory, where $\mathfrak{H}_{\mathrm{int}} = \frac{\lambda}{4!}:\Phi^4:$. Each polynomial T-product vacuum expectation value is evaluated with Wick's theorem - we can forget about the normal ordered parts, as they do not contribute. The best way of bookkeeping is by means of graphs: the pionts $x_1,\dots,x_n$ are to be connected by *propagators* $\overline{\Phi\Phi}$ and *vertices*:



The first term here is zeroth order in λ; not in the figure are the other possible 'disconnected' combinations. The second term is first order in λ and comes with a factor of 4!, since there are 4! different possibilities to connect the lines to the vertex. The third term has a factor of $(4!\cdot 4!/2)\cdot 2!$: two

vertices, hence $(4!)^2$, but exchanging the two inner lines is an automorphism and should hence not be counted; the 2! is for the possibility of exchanging y_1 and y_2.



These are further graphs of order λ^2, except for the last one, which is λ^3. Considering a general graph, we see from the above that when calculating the statistical factor for exchanging vertices, we have to reckon with the following:

- Exchanging vertices (figures 7.2.1 and 7.2.1): this yields topologically different graphs in the first case (fig. 7.2.1), but not in the second case (fig. 7.2.1). So, we have 2 2-automorphisms, yielding a factor $\frac{1}{2} \cdot \frac{1}{2}$.

- Exchanging lines (fig. 7.2.1): this yields topologically identical graphs. Note that the vertex exchange in fig. 7.2.1 can also be viewed as an exchange of the lines connected to the vertices y_3 and y_4.

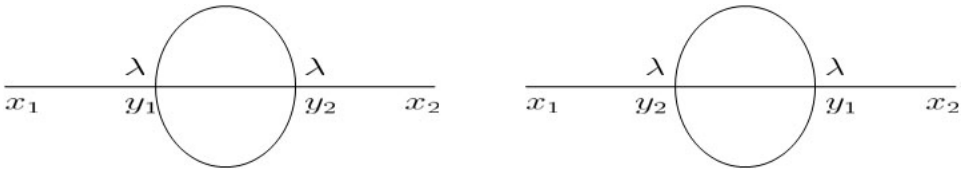


Figure 7.2: Vertex exchange (1)

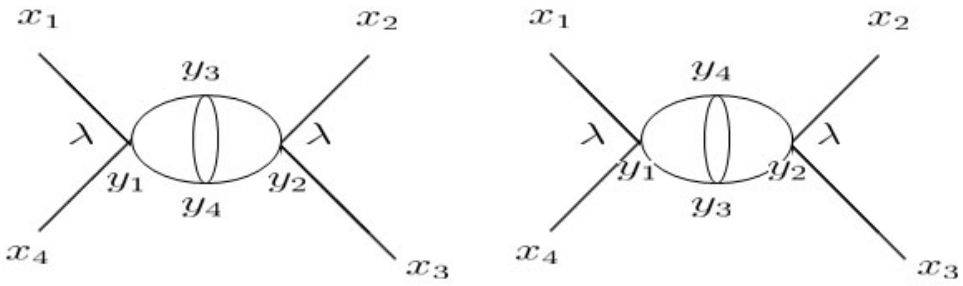


Figure 7.3: Vertex exchange (2)

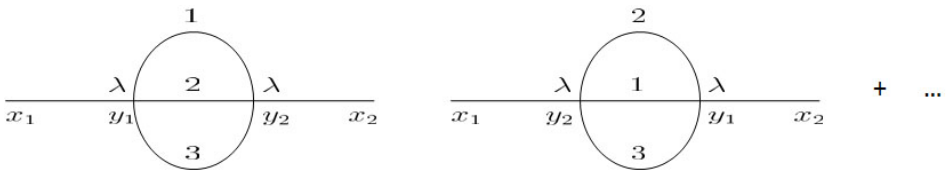


Figure 7.4: Line exchange

In case of doubt, counting the number of possible contractions is a way out. This is more cumbersome than drawing the graphs, but also more reliable.

The factor $\frac{1}{4!}$ in $\frac{\lambda}{4!}$ drops out, since one has to sum over all combinations of the four lines connecting to the rest of the graph. Similarly, in the Gell-Mann-Low formula, the factor $\frac{1}{n!}$ in the exponential is cancelled by the $n!$ ways to arrange the vertices. In loops, there are combinatorial factors correcting the fact that situations are counted several times: one has to divide by the number of automorphisms $\alpha(\Gamma)_{graph}$ of inner lines and vertices which leave the graph unchanged.

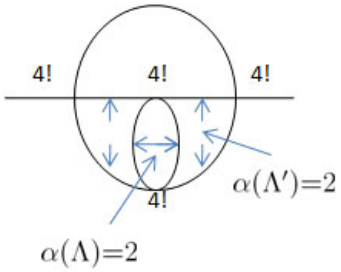


Figure 7.5: Automorphisms

The free propagator $\overline{\Phi\Phi} = \langle 0|\, \mathrm{T}(\Phi_{\mathrm{in}}(x_i)\Phi_{\mathrm{in}}(x_k))\, |0\rangle$ has the form

$$\boxed{\overline{\Phi\Phi} = D_{\mathrm{F}}(x_i - x_k) = \int_{C_{\mathrm{F}}} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x_i - x_k)}} \tag{7.9}$$

where C_{F} is the Feynman path in the imaginary k_0-plane (see fig. 7.1).

Remarks

- for the *interacting* field Φ,

$$\langle 0|\, \mathrm{T}(\Phi(x)\Phi(y))\, |0\rangle = Z D_{\mathrm{F}}(x-y,m) + \int_{>m^2}^{\infty} dm'^2 \sigma(m'^2) D_{\mathrm{F}}(x-y,m') \tag{7.10}$$

one has a prefactor Z (derivation like in chapter 5).

- We will soon see that in the reduction formula for **S**-matrix elements, the first part contributes.

- The normal ordering in $\mathfrak{H}_{\text{int}}$ ensures there are no "tadpole" graphs:

Figure 7.6: Tadpole graphs

There are both connected and "vacuum" graphs in the numerator, but only vacuum graphs in the denominator.

Let us say there are p vertices in the numerator expansion: s of these are connected to outer fields, and the other $p-s$ are in vacuum graphs. Decomposing the graph into its connected and vacuum parts,

$$\frac{(-i)^p}{p!}\langle 0|\,\mathrm{T}(\Phi_{\text{in}}(x_1)\dots\Phi_{\text{in}}(x_n)\mathfrak{H}_{\text{int}}(y_1)\dots\mathfrak{H}_{\text{int}}(y_s))\,|0\rangle\times$$
$$\frac{p!}{s!(p-s)!}\langle 0|\,\mathrm{T}(\mathfrak{H}_{\text{int}}(y_{s+1}\dots\mathfrak{H}_{\text{int}}(y_p))\,|0\rangle$$

we see that the factors of $p!$ cancel each other. This leaves us with

$$\sum_{p,s}\cdots=\sum_{s,p-s}\cdots=\underbrace{\sum_s\frac{1}{s!}\langle\dots\rangle}_{\text{no vac. graphs}}\underbrace{\sum_{p-s}\frac{1}{(p-s)!}\langle\dots\rangle}_{\text{vac. graphs}}$$

- The denominator cancels the vacuum graphs in the numerator.

We thus obtain the *Feynman rules in x-space* for *Wightman functions* $\langle 0|\,\mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle$:

(i) draw all graphs without vacuum parts to some order in the vertices s

(ii) a factor $D_{\rm F}(x)$ for each *propagator* connecting outer points $x_1, \ldots, x_n$ and inner points (vertices) $y_1, \ldots, y_n$; x is the difference between endpoints (here the sign does not matter, but later it will represent the direction of the propagator for charged particles)

(iii) a factor $-i\lambda$ for each vertex (coupling

(iv) combinatorial factors $\frac{1}{\alpha(\Gamma)}$

(v) integrate over the inner points y_i: $\int d^4 y_i$

7.2.2 Feynman rules in momentum space

It is also very useful to Fourier transform the function in the outer x_j, i.e. to perform the integrations

$$\int d^4 x_j e^{-ik_j x_j} \ldots$$

with the outer momenta k_j. Writing down a Fourier representation of $D_{\rm F}$, where outer points and inner vertices are on the same footing,

$$D_{\rm F}(x) = i \int \frac{d^4 x}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon} \tag{7.11}$$

one can perform the x_j and y_i integrations and obtain $(2\pi)^4 \delta^4(k_i, \ldots)$. This gives 4-momentum conservation at the vertices and overall momentum conservation. We can perform the integrations at the vertices, which leaves us with *one* integration per loop and overall momentum conservation. This gives the *Feynman rules in momentum space*:

(i) draw all graphs without vacuum parts to some order in the vertices s

(ii) a factor $\frac{i}{k^2 - m^2 + i\epsilon}$ for each propagator

(iii) a factor $-i\lambda$ for each vertex

(iv) combinatorial factors $\frac{1}{\alpha(\Gamma)}$

(v) integrate over the loop momenta: $\int \frac{d^4 k_i}{(2\pi)^4}$, and add a factor $(2\pi)^4 \delta^4$ for outer momentum conservation

Remarks

- We still have disconnected graphs, and in particular lines running through without interacting. This is where the **S**-matrix comes in, where we have already taken out these cases while applying the reduction formula.

- The vacuum graphs cancel in *our* case, so here they are not very interesting. They contain an overall factor $\delta^4(0_\mu) = \int d^4x e^{ix^\mu \cdot 0_\mu}$, i.e. an infinite volume element. This is related to the fact that these graphs can be shifted in space independently of each other. If there are several disconnected vacuum pieces, they will appear as a sum in the exponential:

$$\langle 0| \dots |0\rangle_{\text{vac}} = \exp\left(\sum \langle 0| \text{ connected vacuum pieces } |0\rangle\right)$$

Exercise: prove this. Hint: use $(a+b)^n/n! = \sum_k \frac{a^k}{k!}\frac{b^{n-k}}{(n-k)!}$.

7.2.3 Feynman rules for complex scalars

For complex scalars, we have

$$\mathbf{H}_{\text{int}} = \frac{\lambda}{4}(\Phi^\dagger(x)\Phi(x))^2 \tag{7.12}$$

The only propagators we have are of the form

$$\langle 0| \,\mathrm{T}(\Phi(x)\Phi^\dagger(y)) \,|0\rangle = D_{\mathrm{F}}(x-y) \tag{7.13}$$

with D_{F} as in eq. (7.9). The other two possibilities give zero:

$$\langle 0| \,\mathrm{T}(\Phi\Phi) \,|0\rangle = \langle 0| \,\mathrm{T}(\Phi^\dagger\Phi^\dagger) \,|0\rangle = 0$$

Since we have complex scalar particles here, we need to change the rules: *particles* propagate from y to x, whereas *anti*particles propagate from x to y. Therefore, an arrow is needed to indicate the direction of propagation. Note that this is *not* related to momentum. See fig. 7.2.3 for a graphical explanation. Also, the factor $\frac{1}{4}$ cancels the number of possibilities to take two $\Phi^\dagger$ and two Φ out of $\mathbf{H}_{\text{int}}$. Each vertex has two incoming lines, corresponding to incoming particles or outgoing antiparticles, and two outgoing ones, corresponding to outgoing particles or incoming antiparticles. In momentum space, p_0 has to be positive for particles and antiparticles, meaning the momentum arrows are different from the particle/antiparticle arrows.

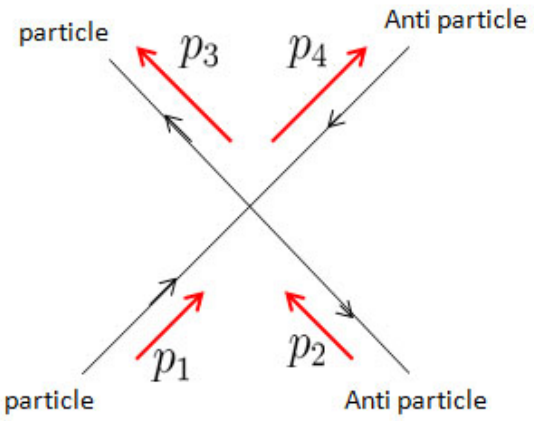


Figure 7.7: Particles and antiparticles with momenta

7.3 Functional relations

As already indicated before, Wightman functions $\langle 0|\,\mathrm{T}(\Phi(x_1),\ldots\Phi(x_n))\,|0\rangle$ with interacting fields Φ can be expanded via the generating functional

$$\boxed{Z(j)=\langle 0|\,\mathrm{T}\{\exp(i\int d^4x\Phi(x)j(x))\}\,|0\rangle} \tag{7.14}$$

by a series of partial differentiations $(\frac{1}{i})^n\frac{\delta Z}{\delta j(x_1)\ldots\delta j(x_n)}|_{j=0}$. In the Gell-Mann-Low formula, the T-products of interacting fields are expressed by those of Φ_{in}-fields, including the **S**-matrix $\mathrm{T}\{\exp[-i\int_{-\infty}^{\infty}d^4y\mathfrak{H}_{\mathrm{int}}(\Phi_{\mathrm{in}}(y))]\}$. Thus, we obtain

$$\langle 0|\,\mathrm{T}(\Phi(x_1)\ldots\Phi(x_n))\,|0\rangle=\left(\frac{1}{i}\right)^n\times \tag{7.15}$$
$$\left.\frac{\frac{\delta}{\delta j(x_1)\ldots\delta j(x_n)}\exp[-i\int d^4y\mathfrak{H}_{\mathrm{int}}\left(\frac{1}{i}\frac{\delta}{\delta j(y)}\right)]\,\langle 0|\,\mathrm{T}\exp\left(i\int d^4x\Phi_{\mathrm{in}}(x)j(x)\right)|0\rangle}{\langle 0|\,\mathrm{T}\exp\left(i\int d^4x\Phi_{\mathrm{in}}(x)j(x)\right)|0\rangle}\right|_{j=0}$$

This gives vacuum graphs with interaction $\mathfrak{H}_{\mathrm{int}}^{(j)}(x)=-\Phi_{\mathrm{in}}(x)j(x)$, which exponentiate to

$$\exp\left(-\frac{1}{2}\int d^4x_1d^4x_2j(x_1)D_{\mathrm{F}}(x_1-x_2)j(x_2)\right)$$

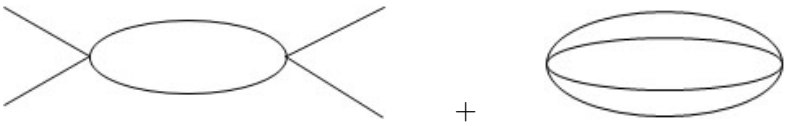
which, with

$$\mathfrak{H}_{\mathrm{int}}\left(\frac{1}{i}\frac{\delta}{\delta j(y)}\right)=\frac{\lambda}{4!}\left(\frac{\delta}{\delta j(y)}\right)^4$$

reproduces the Feynman rules of Φ^4-theory. **Exercise**: show that from the term $(\Phi j)^{2n}/(2n)!$, one obtains

$$\frac{D_{\rm F}^n(2n-1)(2n-2)\dots}{(2n)!} = \left(\frac{D_{\rm F}}{2}\right)^n /n!$$

Graphs with through-going lines are special cases of *disconnected* diagrams. The latter can be decomposed into *connected* diagrams by definition; connected diagrams have all outer lines connected and no separable vacuum graph pieces present (so unseparable vacuum graphs are also called connected). An example of a disconnected graph:



The generator $W(j)$ of such diagrams is related to $Z(j)$ by

$$\begin{aligned} Z(j) &= e^{iW(j)} \quad \text{or} \\ W(j) &= -i\ln Z(j) \end{aligned} \tag{7.16}$$

for normalized $Z(j) \to \frac{Z(j)}{Z(0)}$. Now,

$$\left.\frac{\delta W}{\delta j(x_1)\dots\delta j(x_n)}\right|_{j=0} = i\tau(x_1,\dots,x_n) \tag{7.17}$$

corresponds to connected diagrams with n outer lines.

Note

- For a graph without any outer lines, use $W(0)$. Sum of "connected vacuum diagrams": parts cannot be taken apart. This is the case considered before.

7.4 Back to S-matrix elements

7.4.1 2-point functions

Now, let us consider the Wightman function, also called *2-point function* or *dressed propagator*, for interacting fields Φ. The vacuum expectation value $\langle 0|\,\mathrm{T}(\Phi(x)\Phi(y))\,|0\rangle$, in Φ^4-theory, can be expanded in the language of the Feynman-graphs as depicted in fig. 7.4.1.

The first row gives $D_{\rm F}$, for the first graph, and $D_{\rm F}(-i\Pi)D_{\rm F}$ for the other graphs, where Π is the self-energy or vacuum polarization. These are the

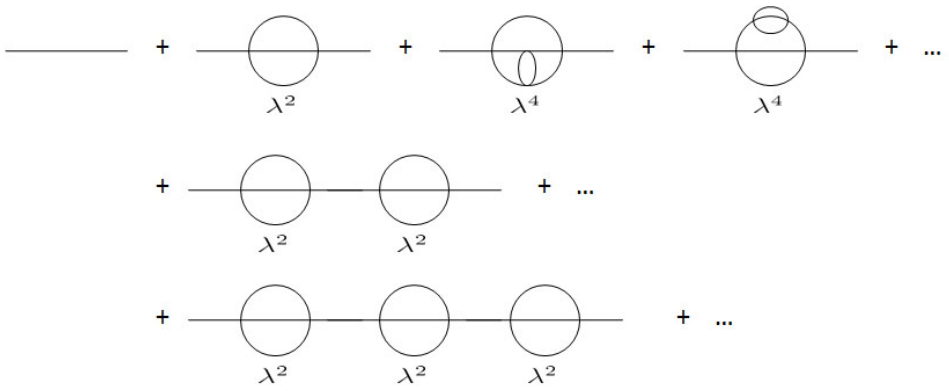


Figure 7.8: Feynman graph representation of $\langle 0|\,\mathrm{T}(\Phi(x)\Phi(y))\,|0\rangle$

so-called *one-particle irreducible* graphs, which cannot be further reduced to two-point graphs by cutting just one inner line. The other rows give a geometrical series:

$$\begin{gathered} D_{\mathrm{F}} + D_{\mathrm{F}}(-i\Pi)D_{\mathrm{F}} + D_{\mathrm{F}}(-i\Pi)D_{\mathrm{F}}(-i\Pi)D_{\mathrm{F}} + \cdots = \\ D_{\mathrm{F}}(1 + i\Pi D_{\mathrm{F}})^{-1} = (D_{\mathrm{F}}^{-1} + i\Pi D_{\mathrm{F}} D_{\mathrm{F}}^{-1})^{-1} = (D_{\mathrm{F}}^{-1} + i\Pi)^{-1} \end{gathered} \tag{7.18}$$

For Fourier transformed 2-point functions, with relative and overall momentum conservation, this reads

$$\left[\left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^{-1} + i\Pi(p^2)\right]^{-1} = \frac{i}{p^2 - m^2 - \Pi(p^2) + i\epsilon} \tag{7.19}$$

The dressed propagator does not contain a particle pole at $p^2 = m^2$ anymore: the denominator now vanishes at $p^2 = m^2 + \Pi(p^2)$, moving the pole to some other m^2. We rename the original $m^2 \to m_0^2$, called "Lagrange mass" or "bare mass", and consider the *physical mass* $m^2 = m_0^2 + \Pi(m^2)$. This means we have to assume a real Π, which will be discussed later. Expanding $\Pi(p^2)$ around $p^2 = m^2$, i.e. around the physical mass, we get

$$\begin{gathered} \frac{i}{p^2 - m_0^2 - (\Pi(m^2) + (p^2 - m^2)\Pi'(m^2) + \Pi_{\mathrm{rest}}) + i\epsilon} = \\ \frac{i}{p^2 - m^2 - (p^2 - m^2)\Pi'(m^2) + \Pi_{\mathrm{rest}} + i\epsilon} = \\ \frac{i(1 - \Pi'(m^2))^{-1}}{p^2 - m^2 - \Pi_{\mathrm{rest}}/(p^2 - m^2) + i\epsilon} \end{gathered} \tag{7.20}$$

As $p^2 \to m^2$, $\Pi_{\mathrm{rest}}/(p^2 - m^2)$ goes to zero, so

$$\lim_{p^2 \to m^2} \frac{i(1 - \Pi'(m^2))^{-1}}{p^2 - m^2 - \Pi_{\mathrm{rest}}/(p^2 - m^2) + i\epsilon} = \frac{iZ}{p^2 - m^2 + i\epsilon}$$

with $Z = (1 + \Pi'(m^2))^{-1}$. This Z is the same as in the LSZ-formalism from chapter 5.

7.4.2 General Wightman functions

In general Wightman functions

$$\langle 0| \mathrm{T}(\Phi(x_1) \dots \Phi(x_n)) |0\rangle$$

the outer lines of the corresponding Feynman graphs also contain these self-energy subgraphs, each with their own inner lines depending on the order of perturbation theory to which one goes. So, in the Fourier transformed expression we have $\frac{iZ}{p^2 - m^2 + i\epsilon}$ for $p^2 \to m^2$.

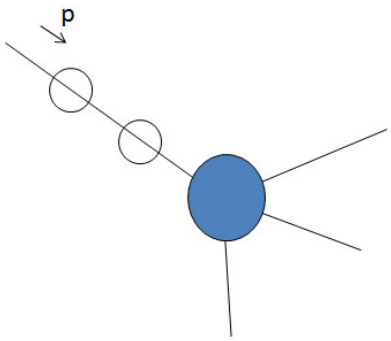


Figure 7.9: Self-energy of outer lines

Now, let us substitute the Fourier transformed vacuum expectation values into the LSZ reduction formula. With only one incoming q_1, we have

$$\frac{i}{\sqrt{Z}} \int d^4x_1 e^{-iq_1x_1} (\partial_{x_1} + m^2) \int \frac{d^4\bar{q}_1}{(2\pi)^4} e^{i\bar{q}_1x_1} F(\langle 0| \mathrm{T}(\dots) |0\rangle) =$$
$$\frac{i}{\sqrt{Z}} \int \frac{d^4\bar{q}_1}{(2\pi)^4} (-\bar{q}_1^2 + m^2)(2\pi)^4 \delta^4(q_1 - \bar{q}_1) \frac{iZ}{\bar{q}_1^2 - m^2 + i\epsilon + \Pi_{\mathrm{rest}}/(\bar{q}_1^2 - m^2)} \tag{7.21}$$

Since the outer particle q_1 is on the mass shell, the only pole is at $\bar{q}_1^2 = q_1^2 = m^2$. Therefore, eq. (7.21) condenses to $\sqrt{Z}$. We will discuss ways to get rid of this remaining factor in the context of renormalization theory, by renormalizing the field Φ (this is known as wave function renormalization).

Note

$-i\Pi(p^2)$ is a divergent quantity in continuum QFT. Consider e.g. the tadpole contribution:

$$-i\Pi_{\text{tadpole}} = (-i)\frac{\lambda}{(2\pi)^4}\int d^4k \frac{i}{k^2 - m_0^2 + i\epsilon} = i\frac{\lambda}{(2\pi)^4}\int \frac{d^4k_{\text{Eucl}}}{-(k_{\text{Eucl}}^2 + m_0^2)}$$

The Wick rotation

$$\begin{aligned} x^0 &= -ix^4 \\ k^0 &= ik^4 \\ \vec{k} &= -\vec{k}_{\text{Eucl}} \\ \vec{x} &= \vec{x}_{\text{Eucl}} \end{aligned} \tag{7.22}$$

gives

$$kx = k^0x^0 - \vec{k}\vec{x} = k^4x^4 + \vec{k}_{\text{Eucl}}\vec{x} = (kx)_{\text{Eucl}} \tag{7.23}$$

$$\Pi = \lambda \int \frac{dk^2k^2}{2} d\Omega_4 \frac{1}{k^2 + m_0^2}$$

(It is not outer p^2-dependent in this case). Introducing a "cut off" Λ^2 in the k^2-integration, we get

$$\Pi^{\Lambda^2} = \frac{\lambda}{2}2\pi^2\frac{1}{(2\pi)^4}\int_0^{\Lambda^2} dk^2 \frac{k^2}{k^2 + m^2} = \frac{\lambda}{16\pi^2}\underbrace{\left\{\int_0^{\Lambda^2} dk^2 - m^2\int_0^{\Lambda^2}\frac{dk^2}{k^2+m^2}\right\}}_{\Lambda^2 - m^2\ln\frac{\Lambda^2+m^2}{m^2}}$$

In the general case, there also is a p^2-dependence, which we will discuss later on in detail. Note that if we want to fix a physical mass (m^2), m_0^2 becomes cut off dependent by $m_0^2 + \Pi = m^2$.

7.5 From S-matrix to cross section

7.5.1 Derivation

The **S**-matrix gives the *transition amplitude* from in- to out-states:

$$\mathbf{S}_{f,p_1p_1} = \langle \alpha_{\text{out}} | p_1, p_{2\text{in}} \rangle \tag{7.24}$$

for scattering two incoming particles p_1 and p_2 into a final state $|\alpha\rangle$, where the in-states have fixed momentum, i.e. are plane waves. Now, we take out the trivial part of **S**, where particles are only running through:

$$\boxed{\mathbf{S} = \mathbb{1} + i\mathbf{T}} \tag{7.25}$$

Making energy-momentum conservation explicit, we have

$$\langle\alpha|\,\mathbf{T}\,|p_1,p_2\rangle = (2\pi)^4\delta^4(p_f-p_1-p_2)\,\langle\alpha|\,\mathbf{M}\,|p_1,p_2\rangle \tag{7.26}$$

In the end, we want to calculate the transition probability. In order to handle the δ-functions properly, let us, before squaring, introduce wave packets for incoming states:

$$|p_1\rangle \to \int \widetilde{dp_1} f_1(p_1)\,|p_1\rangle$$

where $|f_1(p_1)|^2$ is peaked arbitrarily strongly around $\bar{p}_1$. Now, we obtain for the transition probability

$$\begin{aligned} W_{\alpha\leftarrow\bar{p}_1,\bar{p}_2} = \int \widetilde{dp}_1\widetilde{dp}_2 \int \widetilde{dp'}_1\widetilde{dp'}_2 (2\pi)^4\delta^4(p'_1+p'_2-p_1-p_2)\times \\ (2\pi)^4\delta^4(p_f-p_1-p_2) f_1(p_1)f_1^*(p'_1)f_2(p_2)f_2^*(p'_2)\times \\ \langle\alpha|\,\mathbf{M}\,|p_1,p_2\rangle\,\langle\alpha|\,\mathbf{M}\,|p'_1,p'_2\rangle^* \end{aligned}$$

With $(2\pi)^4\delta^4(p'_1+p'_2-p_1-p_2) = \int d^4x\,\exp(-ix(p'_1+p'_2-p_1-p_2))$ and $\int \widetilde{dp}_1 f_1(p_1)e^{ixp_1} = \tilde{f}_1(x)$ (and similar for f_1^*, f_2 and f_2^*), this becomes

$$W_{\alpha\leftarrow\bar{p}_1,\bar{p}_2} = \int d^4x |\tilde{f}_1(x)|^2|\tilde{f}_2(x)|^2(2\pi)^4\delta^4(p_f-p_1-p_2)|\,\langle\alpha|\,\mathbf{M}\,|\bar{p}_1,\bar{p}_2\rangle\,|^2$$

Neglecting the variation of the transition amplitude over the wave packet and assuming $\tilde{f}_i(x) = e^{-i\bar{p}_i x}F_i(x)$ with slowly varying $F_i(x)$ (i.e. $f_i(p_i)$ is strongly peaked around $\bar{p}_i$), the transition probability per volume per time interval is

$$\boxed{\frac{dW}{dV\,dt} = |\tilde{f}_1(x)|^2|\tilde{f}_2(x)|^2(2\pi)^4\delta^4(p_f-\bar{p}_1-\bar{p}_2)|\,\langle\alpha|\,\mathbf{M}\,|\bar{p}_1,\bar{p}_2\rangle\,|^2} \tag{7.27}$$

Now, we still need to normalize the probability current for the positive energy solution: it should satisfy the continuity equation

$$i\tilde{f}^*(x)\overleftrightarrow{\partial}_\mu\tilde{f}(x) \sim 2\bar{p}_\mu|\tilde{f}(x)|^2 \tag{7.28}$$

where the f fulfill the Klein-Gordon equation. The $\sim$-symbol is used here, because $f(p)$ is strongly peaked, so $f(x)$ is almost a plane wave.

Now, when we go into actual experiment, we have a target, at rest in the laboratory's frame, which is being bombarded with particles. In the target, the number of particles per volume element is:

$$\frac{dn_2}{dV} = 2\bar{p}_2^0|\tilde{f}_2(x)|^2 = 2m_2|\tilde{f}_2(x)|^2 \tag{7.29}$$

The incident flux of bombarding particles is given by

$$\underbrace{\frac{|\vec{p}_1|}{p_1^0}}_{\text{velocity}} \times \underbrace{2p_1^0|\tilde{f}_1(x)|^2}_{\text{density}} = 2|\vec{p}_1||\tilde{f}_1(x)|^2 \tag{7.30}$$

7.5.2 Scattering amplitudes and cross sections: an example

Consider two scalar fields, one complex (Φ) and one real (ϕ), with the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \partial_\mu\Phi^*\partial^\mu\Phi - V(\Phi,\Phi^*,\phi) \tag{7.31}$$

where

$$V(\Phi,\Phi^*,\phi) = \frac{1}{2}m^2\phi^2 + m_\Phi^2\Phi^*\Phi + \mu m\Phi^*\Phi\phi + \frac{\lambda}{4}(\Phi^*\Phi)^2 \tag{7.32}$$

Remember the formulas for Φ and ϕ:

$$\Phi(x) = \int \widetilde{dk}\left\{a_\Phi(k)e^{-ikx} + b_\Phi^*(k)e^{ikx}\right\} \tag{7.33}$$

$$\phi(x) = \int \widetilde{dk}\left\{a(k)e^{-ikx} + a^*(k)e^{ikx}\right\} \tag{7.34}$$

Exercise: inserting this into the LSZ-reduction formula, the

(i) outgoing (a_Φ) particles produce a factor

$$\int e^{ipx} \times (\vec{\partial}_x^2 + m^2)\,\langle 0|\,\mathrm{T}(\dots\Phi(x)\dots)\,|0\rangle$$

(ii) incoming (a_Φ) particles produce a factor

$$\int e^{-ipx} \times (\vec{\partial}_x^2 + m^2)\,\langle 0|\,\mathrm{T}(\dots\Phi^\dagger(x)\dots)\,|0\rangle$$

(iii) outgoing (b_Φ) antiparticles produce a factor

$$\int e^{ipx} \times (\vec{\partial}_x^2 + m^2)\,\langle 0|\,\mathrm{T}(\dots\Phi^\dagger(x)\dots)\,|0\rangle$$

(iv) incoming (b_Φ) antiparticles produce a factor

$$\int e^{-ipx} \times (\vec{\partial}_x^2 + m^2)\,\langle 0|\,\mathrm{T}(\dots\Phi(x)\dots)\,|0\rangle$$

We want to calculate the scattering amplitude for particles/antiparticles of Φ-type to order $\mu^2 \sim \lambda$ (this is our choice of parameters). Fig. 7.5.2 represents particle-particle scattering; for antiparticle-antiparticle scattering, one has the same graphs with reversed charge transport directions. Fig. 7.5.2 represents particle-antiparticle scattering.

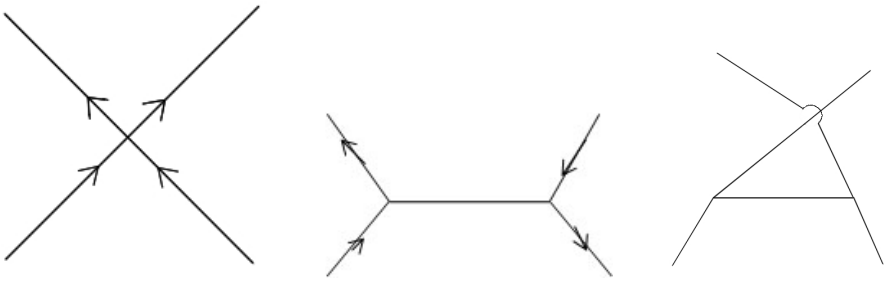


Figure 7.10: Particle-particle Φ^4-interaction

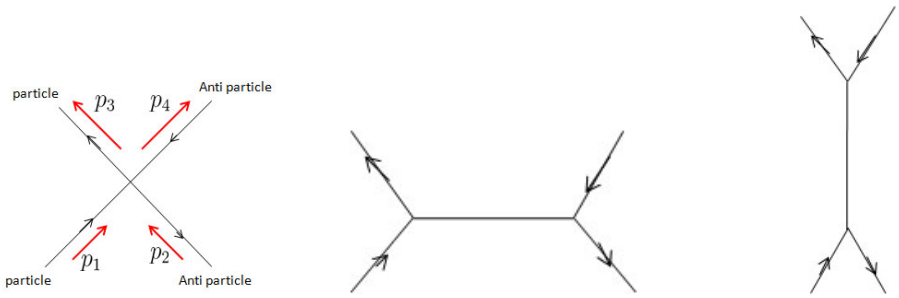


Figure 7.11: Particle-antiparticle Φ^4-interaction

To make the calculations a bit easier, let us introduce the so-called *Mandelstam variables* s, u and t:

$$\begin{aligned}(p_1+p_2)^2 =& s = (p_3+p_4)^2\\ (p_3-p_1)^2 =& t = (p_4-p_2)^2\\ (p_4-p_1)^2 =& u = (p_3-p_2)^2\end{aligned} \tag{7.35}$$

The sum of these three is always $4m_\Phi^2$:

$$s+t+u = 4m_\Phi^2 \tag{7.36}$$

For cases (i) and (ii), the Feynman rules give:

$$\begin{aligned}-i\lambda + (-i\mu m)^2\frac{i}{(p_3-p_1)^2-m^2+\not{i}\epsilon} + (-i\mu m)^2\frac{i}{(p_4-p_1)^2-m^2+\not{i}\epsilon} =\\ -i\left(\lambda + \frac{\mu^2m^2}{t-m^2} + \frac{\mu^2m^2}{u-m^2}\right) = i\mathbf{M}^{(i)}\end{aligned} \tag{7.37}$$

For case (iii), we have

$$-i\left(\lambda + \frac{\mu^2m^2}{t-m^2} + \frac{\mu^2m^2}{s-m^2}\right) = i\mathbf{M}^{(iii)} \tag{7.38}$$

So, exchanging (i) and (iii) comes down to exchanging s and u. Note that we do not need the $i\epsilon$-prescription here, since we do not reach the poles.

Note

Amplitudes are real analytic in two of the three Mandelstam variables; this is called *Mandelstam analyticity* and has been studied extensively in the 1960s. At that time perturbation theory in the case of the strong interactions, what today is the field of quantum chromodynamics (QCD), was considered to be just a guideline to analytic properties of amplitudes, whereas the unitarity of the **S**-matrix, especially in the 2-2 channel, was considered to be most important. See "The analytic **S**-matrix" by Eden, Landshoff, Olive and Polkinghorne (and bootstrap ideas by Chew) for more details.

The amplitudes $\mathbf{M}$ can be used to calculate cross sections with the formulas explained before; in our case, this is particularly simple, since we have only tree-level graphs, i.e. without loops. See the literature on elementary particle physics for more on this topic.

We finally obtain our cross section by calculating

$$\text{(transition probability) / (incident flux} \times \text{target density):}$$
$$d\sigma = (2\pi)^4\delta^4(p_f - p_1 - p_2)\frac{1}{4m_2\bar{p}_1}|\left\langle f\right|\mathbf{M}\left|\bar{p}_1,\bar{p}_2\right\rangle|^2$$

For an n-particle final state in Lorentz covariant notation, this is:

$$d\sigma_{n\leftarrow 2} = \frac{1}{4[(p_1\cdot p_2)^2 - m_1^2m_2^2]^{1/2}}\int \widetilde{dp}_3\ldots\widetilde{dp}_{n+1} \tag{7.39}$$
$$(2\pi)^4\delta^4(p_1+p_2-p_3+p_4-\cdots-p_{n+2})|\left\langle p_3,\ldots p_{n+2}\right|\mathbf{M}\left|p_1,p_2\right\rangle|^2$$

For the fermion case, we also need to sum over, or fix, the spin, but apart from that, the result is the same (this is different from Itzykson and Zuber's book, where the $2m$ factor in $\bar{u}u$ is not introduced; this will be discussed later). For the case with n identical outgoing particles, there will also be a combinatorial factor of $1/n!$.

7.5.3 The optical theorem

Because of probability conservation, $\mathbf{S} = \mathbb{1} + i\mathbf{T}$ is *unitary*:

$$\mathbf{S}^\dagger\mathbf{S} = 1 \tag{7.40}$$

so, $-i(\mathbf{T}-\mathbf{T}^\dagger) = \mathbf{T}^\dagger\mathbf{T}$. Consider

$$\left\langle k_1,k_2\right|\mathbf{T}^\dagger\mathbf{T}\left|k_1,k_2\right\rangle = \sum_n\prod_{i=1}^n\int\widetilde{dq}_i\left\langle k_1,k_2\right|\mathbf{T}^\dagger\left|\{q_i\}\right\rangle\left\langle\{q_i\}\right|\mathbf{T}\left|k_1,k_2\right\rangle$$

with $\{q_i\}$ a complete set of states. Taking out $(2\pi)^4\delta^4(...)$, we obtain a relation for **M**-matrix elements:

$$2\Im(\mathbf{M}(k_1,k_2\to k_1,k_2)) = \sum_n \prod_{i=1}^{n} \int \widetilde{dq_i}(2\pi)^4\delta^4(k_1+k_2-\sum_i q_i)\times \\ |\mathbf{M}(k_1,k_2\to\{q_i\})|^2 \tag{7.41}$$

The second factor of $(2\pi)^4\delta^4(\dots)$ has been taken out here on both sides as $(2\pi)^4\delta^4(k_1+k_2-(k_1+k_2))$. This relates the imaginary part of the forward scattering amplitude to the total cross section. This can be seen explicitly for Feynman diagrams (Cutkosky cutting rules): [picture] Again, see Eden, Landshoff, Olive and Polkinghorne, *The analytic* **S**-*matrix* for more details.

Remark

We have done scattering theory rather superficially - of course this can be done better. See e.g. Weinberg (literature list) for a more elaborate treatment.

Cutkosky rules in short

$$\boxed{\frac{1}{x\pm i\epsilon} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x)} \tag{7.42}$$

Under the integral $\int dx$, P is the *principal value*, obtained by averaging over both possibilities to go around $x=0$. The Feynman-propagator is given by

$$\frac{1}{p^2-m^2+i\epsilon} = P\left(\frac{1}{p^2-m^2}\right) - 2\pi i\delta(p^2-m^2) \tag{7.43}$$

$$\mathbf{M} = \frac{1}{2}\frac{(i\lambda)^2}{i}\int \frac{d^4k_1}{(2\pi)^4}\frac{i}{k_1^2-m^2+i\epsilon}\frac{i}{(p_1+p_2-k_1)^2-m^2+i\epsilon} \tag{7.44}$$

$$\Im(\mathbf{M}) = \frac{(\lambda)^2}{2}\int \frac{d^4k_1}{(2\pi)^4}\delta(k_1^2-m^2)\delta((p_1+p_2-k_1)^2-m^2)(2\pi)^2 = \\ \frac{(\lambda)^2}{2}\int \frac{d^3k_1}{2k_{1_{0,+}}(2\pi)^3}\int \frac{d^3k_2}{2k_{2_{0,+}}(2\pi)^3}(2\pi)^4\delta^4(k_1+k_2'-p_1-p_2)$$

Now, the question is whether k_1 and k_2 are positive. Let us check:

$$\frac{1}{p^2-m^2+i\epsilon} = \frac{1}{p_0-(\vec{p}^2+m^2)^{1/2}+i\epsilon}\times\frac{1}{p_0+(\vec{p}^2+m^2)^{1/2}-i\epsilon}$$

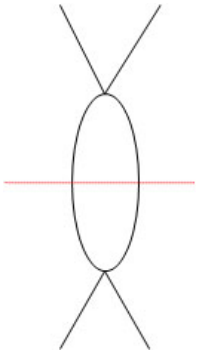


Figure 7.12: Cutting graphs

as discussed before. Integrating around the poles as in the figure *[picture]*, we get from the part around $p_0 = +(\vec{p}^2 + m^2)^{1/2}$

$$\text{(i)} = \left(P\left(\frac{1}{p_0 - (\vec{p}^2 + m^2)^{1/2}} \right) - i\pi\delta(p_0 - (\vec{p}^2 + m^2)^{1/2}) \right) \frac{1}{p_0 + (\vec{p}^2 + m^2)^{1/2}} = P\left(\frac{1}{p^2 - m^2} \right) - 2\pi i\delta(p^2 - m^2)$$

and from $p_0 = -(\vec{p}^2 + m^2)^{1/2}$, we get

$$\text{(ii)} = \left(P\left(\frac{1}{p_0 + (\vec{p}^2 + m^2)^{1/2}} \right) + i\pi\delta(p_0 + (\vec{p}^2 + m^2)^{1/2}) \right) \frac{1}{p_0 - (\vec{p}^2 + m^2)^{1/2}} = P\left(\frac{1}{p^2 - m^2} \right) - 2\pi i\delta(p^2 - m^2)$$

Closing around the poles at $k_{1_0} = (\vec{k}_1^2 + m^2)^{1/2}$,

$$k_{1_0} = p_{1_0} + p_{2_0} = + \left((\vec{k}_1 - \vec{p}_1 - \vec{p}_2)^2 + m^2 \right)^{1/2}$$

in the lower half plane in the above integral (eq. (7.44)), we see that k_{1_0} and k_{2_0} are positive. This enables us to write down the Feynman rules in momentum space for **S**-matrix elements; they differ from the previous ones only in that the outer propagator lines are "truncated" and that there is a $\sqrt{Z}$ instead.

Chapter 8

Path integral formulation of QFT

So far, we have studied QFT in the canonical formalism with operator-valued fields, in the Heisenberg representation. Richard Feynman has formulated another representation of both QM and QFT, which is very intuitive and does not use operators. Its mathematical status, however, is still in development. This method is particularly powerful if one wants to quantize gauge theories - this is why it is necessary to discuss it - and it also allows one to derive the Feynman rules very easily, and to discuss problems beyond perturbation theory, although we will not go into the latter here.

8.1 Path integrals in QM

The path integral formulation of QM centers around the transition amplitude for a QM particle from a position $x(t)$ at a time t to a position $x'(t')$ at time t'. It starts from the Heisenberg picture, where the time dependent operators $\mathbf{X}(t)$ and $\mathbf{P}(t)$ have their respective eigenvectors $|x(t)\rangle$ and $|p(t)\rangle$, with time developments

$$\mathbf{X}(t) = e^{i\mathbf{H}(t-t_0)/\hbar}\mathbf{X}(t_0)e^{-i\mathbf{H}(t-t_0)/\hbar} \tag{8.1}$$

$$|x(t)\rangle = e^{i\mathbf{H}(t-t_0)/\hbar}|x(t_0)\rangle \tag{8.2}$$

and similarly for $\mathbf{P}$ and p. The factors of $\hbar$ have been reinserted here for clarity. Note that the sign in eq. (8.2) is opposite to that of the Schrödinger equation. Note also that this equation describes a transition in time, while the position does not change. Finally, in eq. (8.1), $\mathbf{X}(t_0) = \mathbf{X}_\mathrm{S}$; in the Schrödinger picture, t_0 is usually taken to be 0.

Let us start at the end of our discussion of the path integral formulation,

with the result:

$$\boxed{\langle x'(t')|x(t)\rangle = \int \mathcal{D}x\mathcal{D}p \, \exp\left[i\int_t^{t'} d\tau \left\{p(\tau)\frac{dx}{d\tau} - \mathbf{H}(p,x)\right\}/\hbar\right]} \tag{8.3}$$

Here, $\int \mathcal{D}x$ is a *path integral*, in mathematical circles know as *functional integral*, an integral over all possible paths $x(\tau)$ connecting x and x', with $x(t) = x$ and $x(t') = x'$.

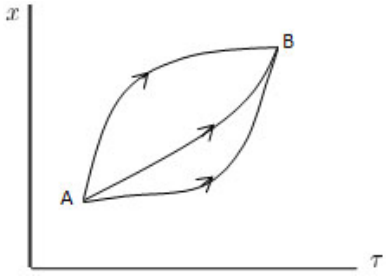


Figure 8.1: Integrate over all possible paths

$\int \mathcal{D}p$ does not have boundary conditions, since the problem asks for the transition amplitude between positions, but not between momenta. (As a small aside: note that the exponent is just the classical action times $i/\hbar$.)

After discretization of the time integral, it becomes a product of integrals at $\tau_1, \tau_2, \ldots$ over $x(\tau_1), x(\tau_2), \ldots$:

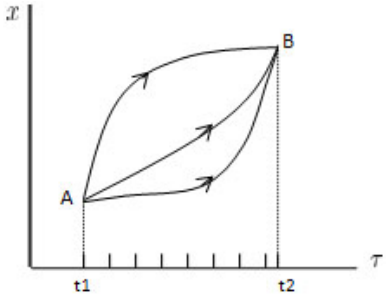


Figure 8.2: Discretization of the time integral

$$\int \mathcal{D}x \exp\left(\int d\tau \ldots\right) \to \prod_i \int_{-\infty}^{\infty} dx(\tau_i)$$

Defining $\delta\tau := \tau_i - \tau_{i-1}$, one eventually has to take the limit $\delta\tau \to 0$, which is mathematically demanding. Obviously, the integral over $\mathcal{D}x$ does not need to be evaluated at the endpoints, since these are fixed.

Now, let us derive this result. From the canonical formalism, we have the following formula:

$$\langle x'(t')|x(t)\rangle = \langle x'|\, e^{-i\mathbf{H}(t'-t)/\hbar}\,|x\rangle \tag{8.4}$$

where $|x\rangle$ is a Schrödinger state. Decomposing the interval into N bits, $t'-t = N\epsilon$, gives

$$\langle x_N|\, e^{-i\mathbf{H}\epsilon/\hbar}\,|x_{N-1}\rangle\,\langle x_{N-1}|\, e^{-i\mathbf{H}\epsilon/\hbar}\,|x_{N-2}\rangle\,\langle x_{N-2}|\dots$$
$$\dots|x_1\rangle\,\langle x_1|\, e^{-i\mathbf{H}\epsilon/\hbar}\,|x_0\rangle$$

where $x_N = x'$ and $x_0 = x$, and $\mathbf{H} = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X})$. Consider one of these matrix elements, to order ϵ:

$$\begin{aligned}
&\langle x_{k+1}|\, e^{-i\mathbf{H}\epsilon/\hbar}\,|x_k\rangle = \langle x_{k+1}|\, 1 - \frac{i\mathbf{H}}{\hbar}\epsilon + \dots|x_k\rangle = \\
&\quad \langle x_{k+1}|x_k\rangle - i\epsilon\int\frac{dp_k}{2\pi\hbar}\left\{V\left(\frac{x_{k+1}+x_k}{2}\right)\langle x_{k+1}|p_k\rangle\,\langle p_k|x_k\rangle + \right. \\
&\qquad \left.\langle x_{k+1}|p_k\rangle\left\langle p_k\left|\frac{\mathbf{P}^2}{2m}\right|x_k\right\rangle\right\} + \dots = \\
&\int\frac{dp_k}{2\pi\hbar}\left[1 - \frac{i\epsilon}{\hbar}\left(\frac{p_k^2}{2m} + V\left(\frac{x_{k+1}+x_k}{2}\right)\right)\right]\exp\left(\frac{ip_k}{\hbar}(x_{k+1}-x_k)\right) + \dots
\end{aligned} \tag{8.5}$$

Note: for more complicated $\mathbf{X}/\mathbf{P}$-mixed operators one needs *Weyl ordering*, a symmetrization of the operator sequence in $\mathbf{X}/\mathbf{P}$; see Peskin & Schröder, p. 281 for more on this topic. Note that the argument in $V(\frac{x_{n+1}+x_n}{2})$ is written like this for cosmetic reasons; we could just as well have written x_n, since in the end, the limit $N\to\infty$ will be taken. Continuing our derivation, let us define

$$\theta(\epsilon) = \int\frac{dp_k}{2\pi\hbar}\exp\left(-\frac{i\epsilon}{\hbar}\left(\frac{p_k^2}{2m} + V\left(\frac{x_k+x_{k+1}}{2}\right) - p_k\frac{x_{k+1}-x_k}{\epsilon}\right)\right) \tag{8.6}$$

which is the right hand side of eq. (8.5) to order $\mathcal{O}(\epsilon)$. Multiplying all θ's and taking the limit $\epsilon\to 0$, we have

$$\langle x'(t')|x(t)\rangle = \int\mathcal{D}x\mathcal{D}p\,\exp\left[i\int_t^{t'}dt\left(\frac{p\dot{x}-H(p,q)}{\hbar}\right)\right] \tag{8.7}$$

with $x(t) = x$ and $x(t') = x'$. This limit is of course accompanied by some higher-level mathematics. The naïve expression, however, has to be based on the discretized version we started from. Concretely, for physicists, this means that in QFT, numerical lattice calculations are an adequate way to approach this integral. Note that the continuous and differentiable functions are a dense set of measure zero in the functional integral.

The dp_k-integration in eq. (8.6) can be performed: it is just a Gaussian integral, here restricted to one dimension for simplicity. It is solved by completing the square:

$$
\begin{aligned}
&-\frac{p_k^2\epsilon}{2m} + p_k(x_k - x_{k+1}) = \\
&-\frac{1}{2}\left[\frac{p_k^2}{m}\epsilon - 2p_k(x_k - x_{k+1}) + \frac{(x_k - x_{k-1})^2}{\epsilon}m\right] + \frac{1}{2}m\frac{(x_k - x_{k-1})^2}{\epsilon} = \\
&-\frac{p_k'^2}{2}\frac{\epsilon}{m} + \frac{1}{2}m\frac{(x_k - x_{k-1})^2}{\epsilon}
\end{aligned}
$$

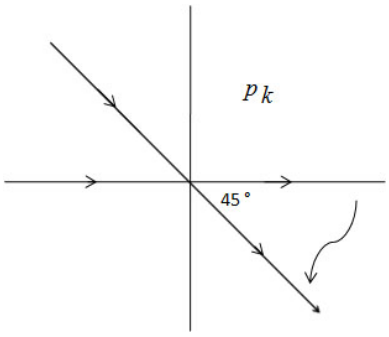
with $p_k' := \frac{p_k}{m} - \frac{x_k - x_{k+1}}{\epsilon}$. Using the standard Gaussian integral,

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \left(\frac{\pi}{\alpha}\right)^{1/2} \tag{8.8}$$

we can perform the p_k'-integral. The first part becomes

$$\int_{-\infty}^{\infty} \frac{dp_k'}{2\pi\hbar} \exp\left(-\frac{1}{2}\frac{i\epsilon}{\hbar m}p_k'^2\right) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy e^{-y^2/2} = \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi\hbar m}{i\epsilon}}$$

where $y = p_k'\sqrt{i\epsilon/\hbar m}$. Note that due to the presence of $\sqrt{i}$ in the conversion from p_k' to y, this substitution constitutes a 45-degree rotation of the integration path:



So, the final result, the product of all the separate integrals, is

$$\prod_{j=1}^{N-1}\int dx_j \left(\sqrt{\frac{m}{i\epsilon\hbar 2\pi}}\right)^N \exp\left(i\sum_{i=1}^{N}\epsilon\left\{\frac{m}{2}\frac{(x_i - x_{i-1})^2}{\epsilon^2} - V(x_i)\right\}/\hbar\right)$$

which, after taking ϵ to zero and N to infinity, becomes:

$$\boxed{\langle x(t)|x'(t')\rangle = \int \mathcal{D}x \exp\left(i\int_t^{t'} d\tau \mathcal{L}(x,\dot{x})/\hbar\right)} \tag{8.9}$$

with singular integration measure ($\sim \epsilon^{-N/2}$) and Lagrangian density $\mathcal{L} = \frac{m\dot{x}^2}{2} - V(x)$.

The x-space path integral is less general (namely only for H quadratic in p) than the first version, but for our purpose, we can settle for this one.

Remarks

- For $\langle x'(t')|\, \mathrm{T}(\mathbf{O}_1(t_1)\mathbf{O}_2(t_2)\dots)\, |x(t)\rangle$, the operators $\mathbf{O}_1, \mathbf{O}_2, \dots$ act in the time slices around $t_1, t_2, \dots$. Then, $\langle x(t_1)|\, \mathbf{O}_1(t_1)\, |x(t_1')\rangle = O_1(x(t_1))\delta(t_1 - t_1')$ where O_1 is a function. Thus, we obtain time-ordering in the path integral, which is decomposed into time slices. This remark is also important if the potential has the shape of a matrix in more complicated settings.
- The oscillating behaviour of the Feynman exponential, which makes the convergence of the integral a more subtle affair, can be avoided if we go to imaginary, or Euclidean, time ($t = x^0 = -ix^4 = -it_{\mathrm{E}}$). This so-called *Wick rotation* helps us to define certain expressions properly. Of course, one has to rotate back at the end of the calculation .

8.1.1 Vacuum expectation values

When going to QFT, we will be interested in vacuum expectation values (cf. correlation functions in statistical physics). Let us briefly investigate them here:

$$\langle 0|\, \mathrm{T}(\mathbf{x}(t_1)\dots\mathbf{x}(t_n))\, |0\rangle = ?$$

Let us start from

$$\langle x_T(T)|\, \mathrm{T}(\mathbf{x}(t_1)\dots\mathbf{x}(t_n))\, |x_{-T}(-T)\rangle = \int_{x_{-T}}^{x_T} \mathcal{D}x\, x(t_1)\dots x(t_n)\times \exp\left(-\int d\tau_e \left(\frac{m\dot{x}^2}{2} + V(x)\right)\right) \tag{8.10}$$

where τ_e stands for Euclidean (Wick-rotated) time. Now,

$$|x_{-T}(-T)\rangle = e^{(-T)H}\, |x_{-T}(0)\rangle = \sum_n |n\rangle\, \langle n|x_{-T}\rangle\, e^{-E_n T}$$

where $|x_{-T}(0)\rangle$ is also written $|x_{-T}\rangle$, and is a Schrödinger vector, which at $t = 0$ coincides with the corresponding Heisenberg vector. When T is large, only the ground state contributes. Then, eq. (8.10) becomes

$$= \langle x_T(0)|0\rangle\, \langle 0|\, \mathrm{T}(\mathbf{x}(t_1)\dots\mathbf{x}(t_n))\, |0\rangle\, \langle 0|x_{-T}(0)\rangle\, e^{-2E_0 T}$$

Dividing by $\langle x_T(T)|x_{-T}(-T)\rangle$, like in the Gell-Mann-Low formula, removes the outer parts, leaving

$$\langle 0|\, \mathrm{T}(\mathbf{x}(t_1)\dots\mathbf{x}(t_n))\, |0\rangle = \lim_{T\to\infty} Z_T \int_{x_{-T}}^{x_T} \mathcal{D}x\, x(t_1)\dots x(t_n) e^{-S} \tag{8.11}$$

where $Z_T = \int_{x_{-T}}^{x_T} \mathcal{D}x\, e^{-S}$ and $S = \int d\tau_e(\frac{m\dot{x}^2}{2} + V(x))$.

Remarks

- Note the similarity with the thermal average in statistical mechanics with the Boltzmann weight $e^{-\beta H}$.
- $x_{\pm\infty}$ is in general not a fixed value; taking the Gell-Mann-Low quotient, the final result should not depend on it.
- The *Feynman-Kac formula*: $\lim_{T\to\infty} \log \langle x_T(T)|x_{-T}(-T)\rangle / (-2T) = E_0$. **Exercise**: derive this result.

8.2 Path integrals in QFT

8.2.1 Framework

Consider real scalar field theory, with $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)$, where $V(\Phi)$ could for example be $\frac{m^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4$. Now, $\Phi(\vec{x}, t)$ substitutes $x_i(t)$ as we go to infinitely many degrees of freedom. The following correspondences hold:

QM	QFT
• $\mathbf{X}_i(t)$ (Heisenberg picture)	• $\Phi(\vec{x}, t)$ Heisenberg operator
• $\lvert x_i(t)\rangle$ state vectors with $\mathbf{X}_i(t)\lvert x_i(t)\rangle = x_i(t)\lvert x_i(t)\rangle$	• Fock space (Φ-eigenvector states; coherent states)
• boundary conditions	• vacuum state $\lvert 0\rangle$ for $t \to \pm\infty$

$|0\rangle$ is unique only without outer fields ("currents"). *With* outer fields (or "currents"), as we have seen before in the generating functional $Z(j)$, we have an extra term $\mathcal{L}_j$ in the Lagrangian density: $\mathcal{L}_j = j(x)\Phi(x)$, where $j \to 0$ for large $\vec{x}$ and t. This implies that $|\Omega_{t\to-\infty}\rangle \neq |\Omega_{t\to\infty}\rangle$ and $|0\rangle_{\text{out}} \neq |0\rangle_{\text{in}}$.

Going from x_i to Φ, our path integral will run over fields Φ, and momentum fields Π, where the latter are given by

$$\Pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi)} = \partial_0\Phi = \partial_t\Phi \tag{8.12}$$

Now, the Φ- and Π-integrals become:

$$\int \mathcal{D}\Phi \to \lim \int \prod_{i,j} d\Phi(\vec{x}_i, \tau_j) = \lim \int \prod_k d\Phi_k$$

$$\int \mathcal{D}\Pi \to \lim \int \prod_k d\Pi_k$$

where the index k represents a 4-dimensional lattice (the limit $\epsilon \to 0$ will be taken in the end). Of these, $\int \mathcal{D}\Pi$ can be performed like in the QM case, and starting at the end again, we obtain:

$$\langle 0|\, \mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle = \frac{\int \mathcal{D}\Phi \exp\left\{i\int d^4x \left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi(x))\right)\Phi(x_1)\dots\Phi(x_n)\right\}}{\int \mathcal{D}\Phi \exp(\dots)} \tag{8.13}$$

($\hbar$ has been set equal to 1 again).

Now, we can sum over all fields with $\Phi(\vec{x},t) \to 0$ as $|\vec{x}|, |t| \to \infty$. (In the quotient, this condition should drop out; strictly speaking, one has to calculate the Fock space vacuum in x-space. See the exercises.) The integrals are, like in QM, well defined after Wick rotation ($x_0 \to -ix_4$ substitution and 90-degree rotation in the x_4-plane) to imaginary time; another option is inserting a term $\frac{1}{2}(m^2 - i\epsilon)\Phi^2(x)$ in the potential. The Wick rotation results in:

$$\partial_\mu\Phi\partial^\mu\Phi - V(\Phi(x)) \to -\partial_\mu\Phi\partial_\mu\Phi - V(\Phi(x_{\mathrm{E}})) \tag{8.14}$$

$$\int d^4x \to -i\int d^4x_{\mathrm{E}} \tag{8.15}$$

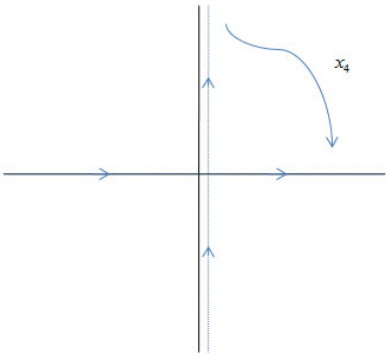


Figure 8.3: Wick rotation

Eq. (8.13) is very similar to the Gell-Mann-Low formula. It can also be written with the help of a source $j(x)$,

$$Z(j) = \int \mathcal{D}\Phi \exp\left(i\int d^4x(\mathcal{L} + j\Phi)\right) \tag{8.16}$$

which gives

$$\langle 0|\,\mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle = \frac{1}{(i)^n Z(0)}\,\frac{\delta^n Z(j)}{\delta j(x_1)\dots\delta j(x_n)}\bigg|_{j=0} \tag{8.17}$$

where the variational derivative provides factors $i\Phi(x_i)$. Note: the normalization of $\int \mathcal{D}\Phi$ is often chosen to give $Z(0) = 1$.

8.2.2 QFT path integral calculations

Deriving eq. (8.13) in a discretized version, like we did for the QM case, involves a multi-component version of the standard Gaussian integral, eq. (8.8):

$$I(A,b) = \prod_{i=1}^{N}\int_{-\infty}^{\infty} d\Phi_i \exp\left(-\frac{1}{2}\Phi_i A_{ik}\Phi_k + b_i\Phi_i\right) \tag{8.18}$$

We diagonalize A by an orthogonal transformation $\mathbf{O}$, whose Jacobi-determinant is 1. Then, we have

$$I(a,b') = \prod_{i=1}^{N}\int_{-\infty}^{\infty} d\Phi'_i \exp\left(-\frac{1}{2}\Phi'_i a_{ii}\Phi'_i + b'_i\Phi'_i\right)$$

Completing the square, we rewrite $-\frac{1}{2}\Phi'_i a_i\Phi'_i + b'_i\Phi'_i$ into $-\frac{1}{2}a_i(\Phi'_i - \frac{b'_i}{a_i})^2 + \frac{1}{2}b'_i\frac{1}{a_i}b'_i$, which gives

$$I(a,b') = \prod_{i=1}^{N}\sqrt{\frac{2\pi}{a_i}}\exp\left(\frac{1}{2}b'_i\frac{1}{a_i}b'_i\right) = (2\pi)^{N/2}\frac{1}{(\det A)^{1/2}}\exp\left(\frac{1}{2}b^T A^{-1}b\right)$$

where the second step is the result of rotating back. So, in the end, we are left with

$$\boxed{I(A,b) = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}}\exp\left(\frac{1}{2}b^T A^{-1}b\right) = I(A,0)\exp\left(\frac{1}{2}b^T A^{-1}b\right)} \tag{8.19}$$

Observe that $\frac{(2\pi)^{N/2}}{(\det A)^{1/2}} = I(A,0)$.

Differentiation with respect to b_i produces vacuum expectation values:

$$\frac{\partial}{\partial b_i}\frac{\partial}{\partial b_k}I(A,b)\bigg|_{b=0} = I(A,0)\times \underbrace{(A^{-1})_{ik}}_{\text{2-point function}} \tag{8.20}$$

Let us apply this to the $Z(j)$ from above, at first without interaction:

$$\mathcal{L} = \partial_\mu\Phi\partial_\mu\Phi + \frac{m^2}{2}\Phi^2 \tag{8.21}$$

Note that this is the Euclidean, i.e. rotated, form. With this $\mathcal{L}$,

$$Z_0^{\mathrm{E}}(j) = Z_0^{\mathrm{E}}(0) \exp\left(\frac{1}{2}\int j(x)(-\partial_{\mathrm{E}}^2 + m^2)^{-1}_{xx'} j(x') d^4x^{\mathrm{E}} d^4x'^{\mathrm{E}}\right) \qquad (8.22)$$

For comparison: in the Minkowski metric, it looks like this:

$$\begin{aligned} Z_0(j) =& Z_0(0) \exp\left(-\frac{1}{2i}\int j(x)(\partial^2 + m^2 - i\epsilon)^{-1}_{xx'} j(x') d^4x d^4x'\right) = \\ & Z_0(0) \exp\left(-\frac{1}{2}\int j(x) D_{\mathrm{F}}(x - x') j(x') d^4x d^4x'\right) \end{aligned}$$

In the Euclidean metric, the calculation is easier due to the absence of the i-factors. It is easy to check that

$$\langle 0|\, \mathrm{T}(\Phi(x_1)\Phi(x_2))\, |0\rangle = \frac{1}{i^2}\, \frac{\delta^2 Z(j)}{\delta j(x_1)\delta j(x_2)}\bigg|_{Z(0)} = D_{\mathrm{F}}(x_1 - x_2)$$

Remarks

- This can also be performed in the Fourier transformed form:

  $$S_0^{\mathrm{E}} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \left(\widetilde{\Phi}(p)(p^2 + m^2)\widetilde{\Phi}(-p) - \widetilde{j}(p)\widetilde{\Phi}(-p) - \widetilde{j}(-p)\widetilde{\Phi}(p)\right)$$

  with $\widetilde{j}(-p) = j^*(p)$ and $\widetilde{\Phi}(-p) = \widetilde{\Phi^*}(p)$, since $\widetilde{j}$ and Φ are real. Redefining

  $$\widetilde{\Phi}(p) = \widetilde{\Phi}'(p) + (p^2 + m^2)^{-1}\widetilde{j}(p)$$

  (i.e., completing the square) gives

  $$Z_0^{\mathrm{E}}(j) = Z_0^{\mathrm{E}}(0) \exp\left(\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, \frac{\widetilde{j}(p)\widetilde{j}(-p)}{p^2 + m^2}\right)$$

- With

  $$W_0(j) = \frac{1}{2}\int dx dx' j(x)(iD_{\mathrm{F}})j(x')$$

  we obtain the following relation for Z and W:

  $$\frac{Z_0(j)}{Z_0(0)} = e^{iW_0(j)} \quad \left(= e^{-W^{\mathrm{E}}(j)}\right)$$

  where the factor iD_{F} generates the connected 2-point functions; this will be generalized later on. This relation resembles the one between the partition function and the free energy in thermodynamics. Note, by the way, that Z and W have been exchanged in the text by Ramond (see literature list).

8.2.3 Perturbation theory

The perturbative approach to path integrals will, again, be introduced in the Euclidean notation. We will be dealing with a potential $V(\Phi)$, which can be written as

$$\boxed{V_{\rm E}(\Phi) \to V_{\rm E}\left(\frac{\delta}{\delta j}\right)} \tag{8.23}$$

where the differentiation is acting on the generating functional $Z_0^{\rm E}(j)$. (For example, $\frac{\lambda}{4!}\Phi^4 \to \frac{\lambda}{4!}\left(\frac{\delta}{\delta j(x)}\right)^4$.) Let us start with

$$\langle 0|\,{\rm T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle = \frac{1}{Z(0)}\frac{\delta^n Z(j)}{\delta j(x_1)\dots\delta j(x_n)}\bigg|_{j=0} \tag{8.24}$$

Now, expand the exponential in the path integral for $Z(j)$ in powers of $V(\Phi)$ and act on $Z_0(j)$ with $V(\frac{\delta}{\delta j})$ as discussed before. Note that also the $\frac{\delta}{\delta j(x_i)}$ in eq. (8.24) act on $Z_0(j)$. The $\frac{\delta}{\delta j}$ of the outer Φ and vertex Φ pairwise remove the j-legs of $\exp(\frac{1}{2}\int jD_{\rm F}j)$. Division by $Z(0)$ eliminates the vacuum graphs (n.b.: $Z_0(0)$ is not the same as $Z(0)$). For example, a vertex with $V = \frac{\lambda}{4!}\Phi^4$ requires 4 propagators (permutation gives the factor of 4!). Also, several vertices, each with their factor of $\frac{1}{n!}$, can be permuted and require a combinatorial factor. From all this, the usual rules for Feynman graphs emerge again:

(i) only graphs with outer lines (no vacuum graphs)

(ii) $D_{\rm F}$ for inner and outer propagators

(iii) λ and $\int d^4y$ for each vertex

(iv) combinatorial factors

Exercise: complex scalar fields:

$$Z(j,\bar{j}) = \int \mathcal{D}\Phi\exp(-\Phi^\dagger\mathbf{A}\Phi + V(\Phi) - \bar{j}\Phi - \Phi^\dagger j)$$

Here, Φ is a complex vector (with index x). Rewriting it as $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ reduces the problem to two real fields $\Phi_{1,2}$. $\mathcal{D}\Phi = \mathcal{D}\Phi_1\mathcal{D}\Phi_2$, or, more elegantly, $\mathcal{D}\Phi\mathcal{D}\Phi^\dagger$ with independent $\Phi, \Phi^\dagger$. $\mathbf{A} = -\partial_{\rm E}^2 + m^2$ in case of the free complex Klein-Gordom field, and $Z_0(j,\bar{j}) = \exp -\bar{j}\mathbf{A}^{-1}j)$. Derive the Feynman rules for $V(\Phi) = \lambda\frac{(\Phi^\dagger\Phi)^2}{4}$.

Recapitulation

In the path integral formulation of QFT, vacuum expectation values of time ordered products of operators are calculated with the following formula:

$$\langle 0|\,\mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle = \frac{1}{Z(0)}\,\frac{\delta^n Z(j)}{\delta j(x_1)\dots\delta j(x_n)}\bigg|_{j=0} \tag{8.25}$$

where

$$Z(j) = e^{-V(\frac{\delta}{\delta j})}Z_0(j) \quad \text{and}$$
$$Z_0(j) = Z_0(0)\exp\left(\frac{1}{2}\int dxdx' j(x)D_{\mathrm{F}}(x-x')j(x')\right) \tag{8.26}$$

- $Z_0(0)$ drops out in eq. (8.25).
- Vacuum diagrams are cancelled by the division by $Z(0)$.
- The Feynman rules are the same as those obtained in the framework of canonical quantization.

Chapter 9

Classification of finite representations of the Lorentz group

9.1 Classification of Lorentz transformations

The Lorentz transformations, like the rotations, form a non-commutative, also called *non-Abelian*, group, which, however, is non-compact. They are represented by matrices Λ with the following property:

$$\boxed{\Lambda^T g \Lambda = g} \tag{9.1}$$

The group operation is defined by executing the transformations sequentially:

$$\begin{aligned} x'' &= \Lambda_2 x' = \Lambda_2 \Lambda_1 x \\ \underbrace{\Lambda_1^T \Lambda_2^T}_{(\Lambda_2 \Lambda_1)^T} g \Lambda_2 \Lambda_1 &= g \end{aligned}$$

Classification

About Lorentz transformations, we know the following:

$$\det(\Lambda^T g \Lambda) = (\det \Lambda)^2 \det g \overset{!}{=} \det g \tag{9.2}$$

So,

$$\boxed{\det \Lambda = \pm 1} \tag{9.3}$$

Furthermore, we know:

$$\Lambda_0^{T\mu} g_{\mu\nu} \Lambda^\nu{}_0 = g_{00} \tag{9.4}$$

In orthogonal coordinates, $(\Lambda^0{}_0)^2 - \sum_k (\Lambda^k{}_0)^2 = 1$, so

$$\boxed{|\Lambda^0{}_0| \geq 1} \tag{9.5}$$

So, in total, we have $2 \times 2 = 4$ possibilities:

- $\det \Lambda = 1$: L_+
- $\det \Lambda = -1$: L_-
- $\Lambda^0{}_0 \geq 1$: $L^\uparrow$
- $\Lambda^0{}_0 \leq -1$: $L^\downarrow$

$L^\uparrow_-$ contains a space inversion, represented by the parity operator $\mathbf{P}$; $L^\downarrow_-$ contains a time reversal, represented by the time inversion operator $\mathbf{T}$:

$$\mathbf{P} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}; \qquad \mathbf{T} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \tag{9.6}$$

$L^\downarrow_+$ contains both time and space inversion, whereas $L^\uparrow_+$ has neither. This last group is called the *proper orthochronous Lorentz group*, which is a subgroup of the Lorentz group.

Now, consider infinitesimal proper orthochronous Lorentz transformations (i.e. continously connected to $\mathbb{1}$):

$$\Lambda = \mathbb{1} + \epsilon M \tag{9.7}$$

M is the generator of these transformations. Applying eq. (9.2) to this Λ, we have $(\mathbb{1} + \epsilon M^T) g (\mathbb{1} + \epsilon M) = g$. Taking this to order $\mathcal{O}(\epsilon)$ gives $M^T g + gM = 0$, or $M^T = -gMg^{-1}$ (remember, $g^{-1} = g$). In orthogonal coordinates,

$$\boxed{M_{ki} = -g_{ii} M_{ik} g_{kk}} \tag{9.8}$$

We will make use of the following vector sets of matrices:

$$\vec{R} := (A_{(2,3)}, A_{(3,1)}, A_{(1,2)}) \qquad \text{antisymmetric} \tag{9.9}$$

$$\vec{B} := (A_{(1,0)}, A_{(2,0)}, A_{(3,0)}) \qquad \text{symmetric} \tag{9.10}$$

with

$$A_{(i,k)} := \begin{cases} (M_{(i,k)})_{ik} = -(M_{(i,k)})_{ki} = -1 \\ \text{zero otherwise} \end{cases} \tag{9.11}$$

The only nonzero entries of $M_{(\alpha,\beta)}$ are $(M_{(\alpha\beta)})_{\alpha\beta}$ and $(M_{(\alpha\beta)})_{\beta\alpha}$. This means that, for example, $(A_{(2,3)})_{kl} = (R_1)_{kl} = -\epsilon_{1kl}$. The R_i are the generators of rotations, whereas the B_i generate boosts. As an exercise, check the following relations:

$$\begin{aligned} [R_i, R_j] &= \epsilon_{ijk} R_k \\ [B_i, B_j] &= -\epsilon_{ijk} R_k \\ [R_i, B_j] &= \epsilon_{ijk} B_k \end{aligned} \tag{9.12}$$

Note that the last commutator tells us that $\vec{B}$ transforms like a vector under rotations.

Boosts

A short reminder: boosts can be considered rotations with an imaginary angle. Consider, for example, a boost in the (0, 1)-plane:

$$\begin{pmatrix} \cosh u & -\sinh u & & 0 \\ -\sinh u & \cosh u & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \overset{\rightarrow}{\text{infin.}} \begin{pmatrix} 1 & -u & & 0 \\ -u & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

where $\tanh u = \beta = v/c$. Boosting from a particle's restframe to one that is moving with velocity v would look like this (remember, $p_0^2 = E^2/c^2 = \vec{p}^2 + m^2c^2$):

$$p = \begin{pmatrix} mc \\ \vec{0} \end{pmatrix} \overset{\text{boost}}{\rightarrow} p = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} mc \\ \vec{0} \end{pmatrix}$$

$$\beta = \frac{v}{c} = \frac{\vec{p}}{E/c}; \qquad \gamma = \underbrace{(1-\beta^2)^{-1/2}}_{\cosh u} = \left(\frac{m^2c^2}{p_0^2}\right)^{-1/2}$$

which gives back the usual relation for boosts. **Exercise**: check the commutation relations.

Note: an arbitrary Lorentz transformation, transforming $p \to p'$, can be represented as

$$\Lambda = B_{p'} D B_p^{-1} \qquad \text{with boost} \qquad \begin{matrix} B_p^{-1} : p \to p_{(0)} \\ B_{p'} : p_{(0)} \to p' \end{matrix} \tag{9.13}$$

where D is the so-called *small group*; in this case, it consists of the rotations. This is called a *Wigner rotation*. Later on, we will come back to this, when discussing *induced representations*.

The commutation relations (9.12) can be reformulated by defining

$$J_l = \frac{1}{2} i(R_l + iB_l) \tag{9.14}$$

$$K_l = \frac{1}{2} i(R_l - iB_l) \tag{9.15}$$

which are Hermitean. (9.12) now become:

$$\begin{aligned} [J_k, J_l] &= i\epsilon_{klm} J_m \\ [K_k, K_l] &= i\epsilon_{klm} J_m \\ [J_k, K_l] &= 0 \end{aligned} \tag{9.16}$$

These are the independent commutators of two sets of generators of the rotation group SO(3) and its covering group SU(2), respectively.

Now, representations are characterized by

$$R^2 - B^2 = -2(\vec{J}^2 + \vec{K}^2) = \frac{1}{2} R_{(\mu,\nu)} M^{(\mu,\nu)} \tag{9.17}$$

$$-\vec{R} \cdot \vec{B} = (\vec{J}^2 - \vec{K}^2) = \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} M_{(\mu,\nu)} M_{(\rho,\sigma)} \tag{9.18}$$

This makes it easy to write irreducible, finite-dimensional representations of the proper orthochronous Lorentz group, such as:

- irreducible representations of infinitesimal rotations, which are classified by (j); $j = 0, \frac{1}{2}, 1, \ldots$
- irreducible representations of infinitesimal $L_+^\uparrow$-transformations, classified by (j, j'); $j, j' = 0, \frac{1}{2}, 1, \ldots$

j is the Casimir operator, whose eigenvalues, which classify the representation, are R^2 and N^2, respectively. However, infinitesimal boosts $N \sim J - K$ generate finite boosts by $e^{n\epsilon N}$ (sic, without i in the power), which is not represented unitarily. However, the action of boosts in representation space is not the same as a transformation in QM Hilbert spacae, as we will see later on.

Remark

Space inversion (parity) $\mathbf{P}$ exchanges J and K:

$$\left.\begin{aligned} \mathbf{P}K &= J\mathbf{P} \\ \mathbf{P}J &= K\mathbf{P} \end{aligned}\right\} \rightarrow \begin{aligned} &\vec{R} \text{ axial vector} \\ &\vec{N} \text{ polar vector} \end{aligned}$$

So, $V^{jj'} \oplus V^{j'j}$ are representations of $L_+^\uparrow + L_-^\uparrow$. The first non-trivial representation is $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, which we will meet again later on, as four-dimensional *Dirac spinor*.

9.2 Poincaré group

We want a general framework for field equations, which is provided by the Poincaré group, also called the *inhomogeneous Lorentz transformations*. It is the analogue of the Galilei group; its generators are P_μ, for translations and $M_{\mu\nu}$, for Lorentz transformations. These operators have the following commutation relations:

$$\begin{aligned} [M_{\mu\nu}, P_\sigma] &= i(g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu) \\ [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}) \end{aligned} \tag{9.19}$$

In a more elegant representation, one uses the *Pauli-Ljubanski vector*:

$$W_\sigma = \frac{1}{2}\epsilon_{\sigma\mu\nu\lambda}M^{\mu\nu}P^\lambda \tag{9.20}$$

It has the following properties:

$$\begin{aligned} W_\mu P^\mu &= 0 \\ \vec{W} &= p_0\vec{R} - \vec{p}\times\vec{N} \\ W_0 &= \vec{p}\cdot\vec{R} \end{aligned}$$

In the restframe of a particle, where $\vec{p}=0$ and $p_0 = m$, the latter gives

$$W_0 = 0 \qquad \text{and} \qquad \vec{W} = m\vec{R}$$

where, in the absence of any momentum, $\vec{R} = \vec{\mathbf{S}}$, the spin.

The Pauli-Ljubanski vector commutes as follows:

$$\begin{aligned} [M_{\mu\nu}, W_\rho] &= i(g_{\nu\rho}W_\mu - g_{\mu\rho}W_\nu) \\ [W_\mu, P_\nu] &= 0 \\ [W_\mu, W_\nu] &= i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma \end{aligned} \tag{9.21}$$

Let us define

$$P^2 = P^\mu P_\mu \qquad \text{and} \qquad W = -W^\mu W_\mu$$

(and note that $W \sim \mathbf{S}^2$ for $m^2 \neq 0$, $\vec{p}=0$). Since P^2 and W commute with all generators, they allow for a *classification of relativistic particles* as unitary representations of the Poincaré group. The complete set of commuting operators consists of:

$$\boxed{P^2,\, W}, \vec{P}, \text{ and } W_3 \text{ or } \frac{\vec{p}\cdot\vec{W}}{p_0|\vec{p}|} = \frac{\vec{p}\cdot\vec{R}}{|\vec{p}|}, \text{ the helicity of the particle}$$

In short, the classification is as follows:

(i) massive particles: $P^2 = m^2$; in the rest frame of such particles, one has $\vec{p} = 0$, $\vec{W} \sim \vec{S}$

(ii) massless particles: $P^2 = 0$, so also $W = 0$. Since we cannot go to the restframe of such a particle, we go to, for example, a frame where $P_\mu^{(0)} = (1, 1, 0, 0)$.

This leads to

$$W_\mu = \lambda P_\mu \quad \text{with} \quad \lambda = \frac{\vec{R} \cdot \vec{p}}{\underbrace{p_0}_{|\vec{p}|}}$$

Now, the commutation relations give

$$\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$$

which correspond to the 1-dimensional (naturally irreducible) representations. For each case, there is a so-called *small group*, which is simply the group of transformations that leaves the standard 4-momenta invariant. In case (i), this group consists of the rotations, in case (ii), the 2-dimensional Euclidean group (rotations and translations in two dimensions) in the plane perpendicular to the helicity.

Remark

Irreducible representations of the small group "induce" irreducible, unitary representations U of the ("noncompact") Lorentz group, which in this case are infinitely-dimensional, unlike before. Consider a normalized state with momentum p and spin σ:

$$\langle p', \sigma' | p, \sigma \rangle = |N(p)|^2 \delta_{\sigma\sigma'} \frac{p_0}{k_0} \delta_3(\vec{p}' - \vec{p})$$

We obtain this state by a boost:

$$|p,\, \sigma\rangle := N(p) \mathbf{U}(L(p)) \, |k,\, \sigma\rangle \tag{9.22}$$

Now, let a unitary representation of a general Lorentz transformation act on it:

$$\begin{aligned} \mathbf{U}(\Lambda) \, |p,\, \sigma\rangle =& N(p) \mathbf{U}(\Lambda L(p)) \, |k,\, \sigma\rangle = \\ & N(p) \mathbf{U}(L(\Lambda p)) \mathbf{U}(L^{-1}(\Lambda p) \Lambda L(p)) \, |k,\, \sigma\rangle \end{aligned}$$

The second $\mathbf{U}$ in the last line sends k to k, but is not necessarily $\mathbb{1}$, and is hence an element of the small group (in e.g. the massive case, a rotation). Let us call it

$$D_{\sigma'\sigma}(W(\Lambda, p)) \left| k,\, \sigma' \right\rangle := \mathbf{U}(L^{-1}(\Lambda p) \Lambda L(p)) \, |k,\, \sigma\rangle \tag{9.23}$$

This allows us to rewrite eq. (9.22) as

$$\mathbf{U}(\Lambda)\,|p,\,\sigma\rangle = \frac{N(p)}{N(\Lambda p)} D_{\sigma'\sigma}(W(\Lambda,p))\,\left|\Lambda p,\sigma'\right\rangle$$

which is the induced infinitely-dimensional unitary representation of the Lorentz group. For further reading, see Wu-Ki Tung, *Group Theory in Physics*.

Remarks

- The finite-dimensional representations of the Lorentz group are *not* unitary. However, field operators $\Psi_\alpha(\vec{x},t)$ do not require direct probability amplitude interpretation, so this is not necessary. We will hear more about this later, in connection with the Dirac equation.

- Remember:

  $$e^{-i\omega t+\vec{k}\cdot\vec{x}} = e^{-i(Et-\vec{p}\cdot\vec{x})/\hbar} = e^{-ip\cdot x/\hbar}$$

  where the exponent $Et-\vec{p}\cdot\vec{x} = \frac{E}{c}ct-\vec{p}\cdot\vec{x} = p_\mu x^\mu = p\cdot x$ is Lorentz invariant.

- A Poincaré transformation $x' = \Lambda x + a$ can be written as a matrix

  $$\begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}$$

  Exercise: show this.

Chapter 10

The Dirac equation and its quantization

10.1 Spinor representation of the Lorentz group

Remember from last chapter:

$$J_l = \frac{i}{2}(R_l + iB_l)$$
$$K_l = \frac{i}{2}(R_l - iB_l)$$

with

$$[J_k, J_l] = i\epsilon_{klm}J_m$$
$$[K_k, K_l] = i\epsilon_{klm}K_m$$
$$[J_l, K_m] = 0$$

That is, we have a representation of the Lie algebra of the Lorentz group as the sum of two Lie algebras of the rotation group SO(3). Keep in mind the distinction between the rotation group SO(3) and its Lie algebra so(3). The Lie algebra is generally notated by lower case so, although capitals (SO)are often used as well.

Recall also the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.1}$$

We have two spinor representations, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, which can be written

with the help of the Pauli matrices, or with the 2×2 versions of J and K:

$$(\tfrac{1}{2},0) \begin{cases} R_l^{(\frac{1}{2},0)} = \frac{J_l+K_l}{i} = -i\frac{\sigma^l}{2} \\ B_l^{(\frac{1}{2},0)} = \frac{J_l-K_l}{-1} = -\frac{\sigma^l}{2} \end{cases}$$

$$(0,\tfrac{1}{2}) \begin{cases} R_l^{(0,\frac{1}{2})} = -i\frac{\sigma^l}{2} \\ B_l^{(0,\frac{1}{2})} = \frac{\sigma^l}{2} \end{cases}$$

Going from $(\frac{1}{2},0)$ to $(0,\frac{1}{2})$, one exchanges the roles of J and K; this is what the space inversion operator $\mathbf{P}$ does, as discussed in last chapter. So, $(\frac{1}{2},0)+(0,\frac{1}{2})$ is an irreducible spinor representation, with

$$R_l^{(\frac{1}{2},0)+(0,\frac{1}{2})} = \begin{pmatrix} -i\sigma^l/2 & 0 \\ 0 & -i\sigma^l/2 \end{pmatrix} \quad \text{and}$$

$$B_l^{(\frac{1}{2},0)+(0,\frac{1}{2})} = \begin{pmatrix} -\sigma^l/2 & 0 \\ 0 & \sigma^l/2 \end{pmatrix} \tag{10.2}$$

General infinitesimal Lorentz transformations are given by

$$\Lambda = \mathbb{1} + \frac{1}{2} M^{(\mu,\nu)} \omega_{\mu\nu} \tag{10.3}$$

(For $M^{(\mu,\nu)}$, see last chapter: $M^{(\mu,\nu)} = -M^{(\nu,\mu)}$; the only non-zero entries of $M^{(\mu,\nu)}$ are $(M^{(\mu,\nu)})_{\mu\nu}$ and $(M^{(\mu,\nu)})_{\mu\nu}$.) Here, $M^{(\mu,\nu)}$ is represented by 4×4-matrices $\lambda^{\mu\nu}$ in spinor space, which must fulfill the commutation relations of the generators of the Lorentz group. If we define the *Dirac γ-matrices* γ^μ by their *anticommutator*

$$\boxed{\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}} \tag{10.4}$$

then

$$\lambda^{\mu\nu} = \frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{10.5}$$

satisfies the commutation relations of the generators of the Lorentz group. **Exercise**: check this.

The choice of γ^μ is not uniquely determined by this requirement, but Pauli has shown all choices to be equivalent. The *Pauli lemma* says that, taking one set of γ^μ, e.g.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{10.6}$$

one can transform to another set $\gamma^{\mu\prime}$ by some transformation $\tilde{S}$:

$$\boxed{\gamma^{\mu\prime} = \tilde{S}\gamma^\mu\tilde{S}^{-1}} \tag{10.7}$$

(Notice that the identity matrices in γ^0 are 2-dimensional.)

Remark

The γ^μ transform like a vector under Lorentz transformations:

$$[\lambda^{\mu\nu}, \gamma^\mu] = -(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}) \tag{10.8}$$

10.2 Spinor fields, Dirac equation

A Dirac spinor transforms as follows:

$$\Psi_\alpha \to S_{\alpha\beta}\Psi_\beta = \Psi'_\alpha \tag{10.9}$$

These transformations are generated by $\lambda^{\mu\nu}\omega_{\mu\nu}$: infinitesimal ones are given by

$$S = \mathbb{1} + \frac{1}{2}\lambda^{\mu\nu}\omega_{\mu\nu}$$

with infinitesimal $\omega_{\mu\nu}$; finite ones come with the full exponential:

$$S = \exp\left(\frac{1}{2}\lambda^{\mu\nu}\omega_{\mu\nu}\right) \tag{10.10}$$

with finite $\omega_{\mu\nu}$. A Dirac spinor field by definition transforms as follows:

$$\boxed{\Psi'_\alpha(x') = S_{\alpha\beta}(\Lambda)\Psi_\beta(x)} \tag{10.11}$$

where $x' = \Lambda x$. This begs the question what the corresponding Lorentz covariant field equation is. To find the answer, begin with remembering that γ^μ transforms like a Lorentz vector:

$$\boxed{S^{-1}\gamma^\mu S = \Lambda^\mu{}_{\mu'}\gamma^{\mu'}} \tag{10.12}$$

and conclude that therefore, if $\Psi(x)$ is a solution, $\Psi'(x)$ must also be one. The sought-for equation, know as the Dirac equation, is

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0} \tag{10.13}$$

or, in components:

$$(i\gamma^\mu_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta})\Psi_\beta(x) = 0$$

For the proof, we will use the following notation:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \qquad \text{and} \qquad \partial_{\mu'} = \frac{\partial}{\partial x'^\mu}$$

The proof goes as follows: assuming $\Psi(x')$ is a solution, and using eq. (10.12), we have

$$\begin{aligned}(i\gamma^{\mu'}\partial_{\mu'} - m)\Psi'(x') =& (i\gamma^{\mu'}\partial_{\mu'} - m)S\Psi(\Lambda^{-1}x') = \\ & (i\gamma^{\mu'}\partial_{\mu}(\Lambda^{-1})^{\mu}{}_{\mu'} - m)S\Psi(x) \stackrel{!}{=} \\ & S(i\gamma^{\mu}\partial_{\mu} - m)\Psi(x) = 0\end{aligned}$$

In the second step, we have rewritten $\partial_{\mu'}$ as $\partial_{\mu}(\Lambda^{-1})^{\mu}{}_{\mu'}$, and in the third one, we have used eq. (10.12) in the form $(\Lambda^{-1})^{\mu}{}_{\mu'}\gamma^{\mu'}S = S\gamma^{\mu}$.

10.3 Construction of the representation matrices in spinor space

In section 10.1, we started the explicit construction of the representation matrices in spinor space. Let us finish this:

10.3.1 Rotations

Consider the following mapping:

$$\boxed{\vec{x} \to \hat{x} = \vec{\sigma}\cdot\vec{x}} = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix} \tag{10.14}$$

Notice that $\hat{x}$ is Hermitean. Further, check that

$$\begin{aligned}\det\hat{x} = -\vec{x}^2 \qquad \text{and} \\ x^i = \frac{1}{2}\mathrm{tr}(\sigma^i\hat{x})\end{aligned} \tag{10.15}$$

(for the last one, use $\frac{1}{2}\mathrm{tr}\sigma^i\sigma^k = \delta^{ik}$). Now, investigate a rotation or inversion (i.e., an element from the orthogonal group) D:

$$\vec{x}' = D\vec{x}$$

with, by definition, $\vec{x}'^2 = \vec{x}^2$ and $D^\dagger D = 1$. Consider a unitary representation

$$\hat{x}' = \mathbf{U}\hat{x}\mathbf{U}^\dagger \tag{10.16}$$

with $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = 1$. Notice that, due to the unitarity of $\mathbf{U}$, we have

$$\det\hat{x}' = \det\hat{x} \qquad \text{or} \qquad -\vec{x}'^2 = -\vec{x}^2 \tag{10.17}$$

In other words, $\mathbf{U}$ represents a rotation or inversion. Without loss of generality, we can choose $\det\mathbf{U} = 1$; this means that

$$\mathbf{U} \in \mathrm{SU}(2)$$

where SU(2) stands for the set of *special, unitary,* 2×2*-matrices*, "special" meaning that its determinant is 1. The general form of $\mathbf{U}$ is

$$\boxed{\mathbf{U} = e^{-i\vec{\sigma}\cdot\vec{\varphi}/2}} \tag{10.18}$$

which is the general rotation operator of QM acting on spin-$\frac{1}{2}$ objects. Infinitesimal transformations are given by

$$\begin{aligned}\hat{x}' =& \hat{x} + \left[-i\frac{\vec{\sigma}}{2}\cdot\vec{\varphi}, \vec{\sigma}\cdot\vec{x}\right] = \\ & \hat{x} + \left(\frac{-i}{2}\right) 2i(\vec{\varphi}\times\vec{x})\cdot\vec{\sigma}\end{aligned} \tag{10.19}$$

10.3.2 Lorentz transformations

For Lorentz transformations, the procedure is very similar. We use the mapping

$$\boxed{x \to \hat{x} = \sigma^0 x^0 - \vec{\sigma}\cdot\vec{x}} = \begin{pmatrix} x^0 - x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 + x^3 \end{pmatrix} \tag{10.20}$$

which again is Hermitean. We also have the 4-dimensional analogues of eqs. (10.15):

$$\det\hat{x} = (x^0)^2 - \vec{x}^2 = x^2 \qquad \text{and} \qquad x^\mu = \frac{1}{2}\mathrm{tr}(\sigma^\mu\hat{x}) \tag{10.21}$$

Now, let us consider the transformation

$$\hat{x}' = M\hat{x}M^\dagger \tag{10.22}$$

with $\det\hat{x}' = \det\hat{x}$, and hence $|\det M|^2 = 1$. Here, M is some arbitrary matrix, for which we can, without loss of generality, assume $\det M = 1$. This implies

$$M \in \mathrm{SL}(2,\mathrm{C}) \tag{10.23}$$

i.e., M is a *special, linear,* 2×2*-matrix with complex entries.* This means it has three complex parameters (or six real ones) to be fixed:

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \text{with} \qquad \alpha\delta - \beta\gamma = 1$$

Of course, $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathrm{C})$.

M acts in the representation space, which in this case is a 2-dimensional spinor space: the $(\frac{1}{2},0)$ representation discussed before. In an abstract vector basis, its action is

$$\chi \to \chi' = M\chi$$

In the basis

$$\left(\begin{array}{c} 1 \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ 1 \end{array} \right)$$

this becomes

$$\chi^s \to M\chi^s = \sum_{s'=-1/2}^{1/2} \chi^{s'} \underbrace{(\chi^{s'} M \chi^s)}_{M_{s's}}$$

10.3.3 Boosts

In order to be able to reformulate the Dirac equation in momentum space, we will need a representation for boosts. Boosting the momentum vector p^μ

$$\left(\begin{array}{c} m \\ \vec{0} \end{array} \right) \overset{\text{boost}}{\to} \left(\begin{array}{c} p^0 \\ \vec{p} \end{array} \right),$$

represented by

$$\hat{p} = m\mathbb{1} \to \hat{p}' = p^0\mathbb{1} - \vec{p}\cdot\vec{\sigma} = M\hat{p}M^\dagger,$$

is achieved with

$$M = L_{\vec{p}} = \frac{p^0 + m - \vec{\sigma}\cdot\vec{p}}{(2m(m+p^0))^{1/2}} \overset{!}{=} L^+_{\vec{p}} \tag{10.24}$$

(As a link to section 10.1: $M = L_{\vec{p}}$ is just $e^{-(\vec{\sigma}/2)\cdot(\vec{p}/|\vec{p}|)u}$ with $\tanh u = \frac{|\vec{p}|}{p^0}$. For $L^+_{\vec{p}}$, chapter 9.) To prove this, first show that

$$\begin{aligned} L_{\vec{p}}\, m\mathbb{1}\, L^+_{\vec{p}} &= p^0\mathbb{1} - \vec{\sigma}\cdot\vec{p} \qquad \text{and} \\ L_{\vec{p}}\, \vec{\sigma}\cdot\vec{p}_\perp L^+_{\vec{p}} &= \vec{\sigma}\cdot\vec{p}_\perp \end{aligned}$$

Boosting in opposite direction is done by means of $L^{-1}_{\vec{p}} = L_{-\vec{p}}$. Using $(a^0\mathbb{1} + \vec{\sigma}\cdot\vec{a})(a^0\mathbb{1} - \vec{\sigma}\cdot\vec{a}) = (a^0)^2 - \vec{a}^2$, we get

$$\begin{aligned} (p^0\mathbb{1} + \vec{\sigma}\cdot\vec{p})L_{\vec{p}} &= mL^{-1}_{\vec{p}} \\ (p^0\mathbb{1} - \vec{\sigma}\cdot\vec{p})L^{-1}_{\vec{p}} &= mL_{\vec{p}} \end{aligned} \tag{10.25}$$

Inverting space by means of the parity operator $\mathbf{P}$ sends $\vec{p}$ to $-\vec{p}$ and

$$L_{\vec{p}}\chi^s \to L_{-\vec{p}}\chi^s = L^{-1}_{\vec{p}}\chi^s$$

Defining

$$u^s_{\vec{p}} = \left(\begin{array}{c} L_{\vec{p}}\chi^s \\ L^{-1}_{\vec{p}}\chi^s \end{array} \right) = \left(\begin{array}{c} \xi \\ \chi \end{array} \right) \tag{10.26}$$

we can rewrite eqs. (10.25) as

$$\boxed{\begin{array}{l}(p^0+\vec{\sigma}\cdot\vec{p})\xi = m\chi \\ (p^0-\vec{\sigma}\cdot\vec{p})\chi = m\xi\end{array}} \tag{10.27}$$

Substituting eqs. (10.27) into each other gives

$$(p^0-\vec{\sigma}\cdot\vec{p})(p^0+\vec{\sigma}\cdot\vec{p})\xi = m^2\xi$$
$$(p^0+\vec{\sigma}\cdot\vec{p})(p^0-\vec{\sigma}\cdot\vec{p})\chi = m^2\chi$$

which leads us to

$$\boxed{(p^0)^2-\vec{p}^2 = m^2} \tag{10.28}$$

which is the mass shell condition. One could, letting go of some mathematical rigour, interpret eqs. (10.27) as the square root of the mass shell condition.

Remarks

If M is a spinor representation of the Lorentz group, so are $(M^\dagger)^{-1}$, $M^* = (M^\dagger)^T$ and $(M^T)^{-1}$. However, since

$$\boxed{M^{-1} = \sigma_2 M^T \sigma_2} \qquad \text{with} \qquad \sigma_2 = i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \sigma_2^{-1} = \sigma_2^\dagger, \tag{10.29}$$

M is equivalent to $(M^T)^{-1}$, and M^* to $(M^\dagger)^{-1}$.

Exercise: check eq. (10.29). Use the general ansatz

$$M = a\mathbb{1} + \vec{b}\cdot\vec{\sigma}, \qquad a^2-b^2 = 1$$

The actions of the four representations are as follows:

M acts on ξ^α

$(M^T)^{-1}$ acts on ξ_α

$(M^\dagger)^{-1}$ acts on $\chi_{\dot\alpha}$

M^* acts on $\chi^{\dot\alpha}$

Letting σ_2 act on ξ_α or $\chi^{\dot\alpha}$ raises or lowers the index; the point indicates which representation is used. The "pointed" representation M corresponds to $(\frac{1}{2},0)$, whereas the "unpointed" $(M^\dagger)^{-1}$ corresponds to $(0,\frac{1}{2})$. The spinor u_p^s consists of a "left-handed" and a "right-handed" component (see below for more on handedness):

$$u_p^s = \begin{pmatrix}\xi^\alpha \\ \chi_{\dot\alpha}\end{pmatrix} \qquad \begin{array}{c}\text{left-handed}\\ \text{right-handed}\end{array}$$

Remark

The vector $\hat{x}$ transforms as $x^{\alpha\dot{\beta}}$: the index α transforms with M, and the $\dot{\beta}$ with $(M^\dagger)^T = M^*$.

10.4 Field equation

10.4.1 Derivation

Now, let us derive the actual field equation. We know that

$$u^s_{\vec{p}} e^{-ipx} = \begin{pmatrix} L_{\vec{p}}\chi^s \\ L_{-\vec{p}}\chi^s \end{pmatrix}$$

should be a solution of a (free) field equation. From eqs. (10.27), we obtain

$$\left\{ p^0 \mathbb{1}_4 - \begin{pmatrix} -\vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix} \right\} u^s_{\vec{p}} e^{-ipx} - m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u^s_{\vec{p}} e^{-ipx} = 0 \quad (10.30)$$

Let us do some renaming, to make it look a little simpler:

$$\vec{\alpha}\cdot\vec{p} := \begin{pmatrix} -\vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix} \qquad \text{and} \qquad \beta := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observe that $\beta^2 = \mathbb{1}$. Furthermore, we will name

$$\beta = \gamma^0 \qquad \text{and} \qquad \beta\vec{\alpha} = \vec{\gamma}$$

(**Exercise**: check that these satisfy the requirements for the Dirac γ-matrices.) Now, multiply eq. (10.30) with β:

$$(\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} - m) \underbrace{u^s_{\vec{p}} e^{-ipx}}_{\Psi(x)} = 0$$

and rewrite it into a differential equation, with $p_\mu = i\partial/\partial x^\mu$:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0} \quad (10.31)$$

This is the Dirac equation. It also has other guises: going back to momentum space, it looks like

$$(\gamma^\mu p_\mu - m)\Psi = 0$$

In the Feynman slash notation, where $\gamma^\mu \partial_\mu = \not{\partial}$ (and similarly $\not{p} = \gamma^\mu p_\mu$), it takes the form

$$(i\not{\partial} - m)\Psi = 0$$

10.4.2 Choosing the γ-matrices

In the derivation above, γ^0 was Hermitean, whereas the γ^i were anti-Hermitean. Since we know $\{\gamma^{\mu\dagger}, \gamma^{\nu\dagger}\} = 2g^{\mu\nu}$, as the $\gamma^{\mu\dagger}$ should be equivalent to γ^μ by the Pauli lemma,

$$\gamma^{\mu\dagger} = \beta\gamma^\mu\beta^{-1} \tag{10.32}$$

(here with β as the γ^0 from section 10.4.1). Taking the Hermitean conjugate of eq. (10.32), we get

$$\gamma^\mu = \beta^{-1\dagger}\gamma^{\mu\dagger}\beta^\dagger$$

Sandwiching this between $\beta^\dagger$ and $\beta^{-1\dagger}$ gives

$$\gamma^{\mu\dagger} = \beta^\dagger\gamma^\mu\beta^{-1\dagger}$$

which, combined with eq. (10.32), allows us to conclude that

$$\beta = \beta^\dagger \tag{10.33}$$

Different representations of the Dirac algebra

The representation used above is called the *chiral* or *Weyl* representation. It has

$${}^W\gamma^0 = \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad {}^W\gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{10.34}$$

(Itzykson and Zuber use $\gamma^0 \to -\gamma^0$.) We can define a fifth γ-matrix:

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{10.35}$$

This has the nice property that $(\mathbb{1} \pm \gamma^5)/2$ projects onto the upper and lower components of spinors. Representing Ψ as $(\Psi_\mathrm{L}\ \Psi_\mathrm{R})^T$, it selects the left- or right-handed component of Ψ:

$$\Psi \stackrel{!}{=} \left(\frac{\mathbb{1}+\gamma_5}{2} + \frac{\mathbb{1}-\gamma_5}{2}\right)\Psi = \left(\frac{\mathbb{1}+\gamma_5}{2} + \frac{\mathbb{1}-\gamma_5}{2}\right)\begin{pmatrix} \Psi_\mathrm{L} \\ \Psi_\mathrm{R} \end{pmatrix} = \begin{pmatrix} \Psi_\mathrm{L} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \Psi_\mathrm{R} \end{pmatrix} = \begin{pmatrix} \Psi_\mathrm{L} \\ \Psi_\mathrm{R} \end{pmatrix} = \Psi$$

Also often used, mainly in the non-relativistic limit of the Dirac equation, is the *Dirac-Pauli representation*:

$${}^D\gamma^0 = \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad {}^D\gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{10.36}$$

In this representation, γ^5 is not diagonal anymore:

$$\gamma^5 = \gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{10.37}$$

The conversion from the Weyl to the Dirac-Pauli representation is as follows:

$${}^D\gamma^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} {}^W\gamma^\mu \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \tag{10.38}$$

Dirac spinors are obtained from their Weyl counterparts by means of the same matrix:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \xi^\alpha \\ \chi_{\dot{\alpha}} \end{pmatrix} \tag{10.39}$$

Remarks

- The Dirac equation is a differential equation of first order in time, just like the Schrödinger equation:

  $$i\partial_0 \Psi = \left(m\beta + \frac{\vec{\alpha} \cdot \vec{\nabla}}{i} \right) \Psi$$

  Notice that both terms between the brackets are Hermitean.

- Weyl spinors ξ^α are very common in GUTs, neutrino physics and supersymmetry. In other fields, the Dirac notation dominates.

10.4.3 Relativistic covariance of the Dirac equation

The fact that the Dirac equation is relativistically covariant is rather obvious, since we have been talking about Lorentz covariant objects all the time. However, let us check it explicitly. Supposing we have a solution $\Psi_\beta(x)$, we want to show that its transformed form

$$\boxed{\Psi'_\alpha(x') = S_{\alpha\beta}(\Lambda)\Psi_\beta(x)} \tag{10.40}$$

is also a solution. In the above equation, as usual, $x' = \Lambda x$ and S is a representation of the Lorentz group $(\frac{1}{2}, 0) + (0, \frac{1}{2})$. If Ψ is written in terms of Weyl spinors, S becomes

$$S = \begin{pmatrix} M & 0 \\ 0 & (M^\dagger)^{-1} \end{pmatrix}$$

The proof that $\Psi'(x)$ also solves the Dirac equation was given at the end of section 10.2; it holds under the condition that

$$(\Lambda^{-1})^\mu{}_{\mu'} \gamma^{\mu'} S = S\gamma^\mu$$

or

$$\boxed{S^{-1}\gamma^{\mu'}S = \Lambda^{\mu'}_{\ \mu}\gamma^{\mu}} \tag{10.41}$$

This corresponds to the transformation property of $\hat{\sigma}$ if we write S in the above form.

Remarks

- Remember that in its infinitesimal form, this S is just the $\mathbb{1} + \frac{1}{2}\lambda^{\mu\nu}\omega_{\mu\nu}$ we have seen before.
- The Dirac versions of $\vec{R}$ and $\vec{B}$ are:
$$\vec{R}^D = i\gamma^5\gamma^0\vec{\gamma}\frac{2}{4}, \qquad \vec{B}^D = i\gamma^0\vec{\gamma}\frac{2}{4}$$
- Space inversion $S(\Pi) = \gamma^0$ exchanges ξ^α with $\chi_{\dot{\alpha}}$.

10.5 Complete solution of the Dirac equation

In deriving the Dirac equation, we began with writing down a solution. However, this is not necessarily a unique one; let us therefore now invert the procedure in order to find all solutions. So far, we have one:

$$\Psi(x) = \frac{1}{(2\pi)^3}u_{\vec{p}}\,e^{-ipx} \qquad \text{with positive } p_0 = \vec{p}^2 + m^2 \tag{10.42}$$

Going to the momentum basis (by means of a Fourier transform), this becomes

$$(\gamma^\mu p_\mu - m)u_{\vec{p}} = 0$$

with

$$\boxed{u^s_{\vec{p}} = \frac{m^{1/2}}{(2m(m+p^0))^{1/2}}\begin{pmatrix} p^0 + m - \vec{\sigma}\cdot\vec{p} \\ p^0 + m + \vec{\sigma}\cdot\vec{p}\end{pmatrix}\chi^s} \tag{10.43}$$

The factor of $m^{1/2}$ is put in by convention (rescaling $u^s_{\vec{p}}$ is always possible). Eq. (10.43) is written in the chiral basis; going to the Dirac-Pauli basis, we will use ${}^D\gamma^\mu = B^W\gamma^\mu B^{-1}$ with

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1}\end{pmatrix} \qquad \text{and} \qquad B^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1}\end{pmatrix} \tag{10.44}$$

In this basis, we have

$$\boxed{Bu^s_{\vec{p}} = \frac{1}{(m+p^0)^{1/2}}\begin{pmatrix} p^0 + m \\ \vec{\sigma}\cdot\vec{p}\end{pmatrix}\chi^s} \tag{10.45}$$

In analogy to the case of the Klein-Gordon equation (chapter 4), we might expect a negative energy solution to exist. Let us make the ansatz

$$\Psi(x) = \frac{1}{(2\pi)^3} v_{\vec{p}}\, e^{ipx} \tag{10.46}$$

In momentum space, this would mean

$$(-\gamma^\mu p_\mu - m) v_{\vec{p}} = 0$$

Now we still need to find out what $v_{\vec{p}}$ actually is. To that end, let us go to another representation of the γ's. First, we define $\mathbf{C}$:

$$\boxed{\mathbf{C}\gamma^{\mu T}\mathbf{C}^{-1} = -\gamma^\mu} \tag{10.47}$$

We know that this $\mathbf{C}$ must exist because of the Pauli lemma: if γ^μ fulfill the requirements, so do $\gamma^{\mu T}$ and $-\gamma^\mu$. It is given by

$$\mathbf{C} = i\gamma^0\gamma^2 \tag{10.48}$$

Exercise: check this. Now, we need only one further definition:

$$\begin{aligned} \bar{\Psi} &:= \Psi^\dagger\beta \qquad (\text{also} \quad \bar{u}^s_{\vec{p}} = u^{s\dagger}_{\vec{p}}\beta) \quad \text{and} \\ \bar{\gamma^\mu} &:= \beta\gamma^{\mu\dagger}\beta \stackrel{!}{=} \gamma^\mu \end{aligned} \tag{10.49}$$

Now, we can simply obtain $v^s_{\vec{p}}$ from the already known solution $u^s_{\vec{p}}$: we begin with

$$(\gamma p - m) u^s_{\vec{p}} = 0$$

and take the "barred conjugate" of this equation:

$$\bar{u}^s_{\vec{p}}(\bar{\gamma} p - m) = 0$$

Then, we take the transpose and multiply that with $\mathbf{C}$:

$$\begin{aligned} \mathbf{C}(\gamma^T p - m)\mathbf{C}^{-1}\mathbf{C}\,\bar{u}^{sT}_{\vec{p}} &= \\ (\mathbf{C}\gamma^T p\mathbf{C}^{-1} - m)\mathbf{C}\,\bar{u}^{sT}_{\vec{p}} &= \\ (-\gamma p - m)\mathbf{C}\,\bar{u}^{sT}_{\vec{p}} &= 0 \end{aligned}$$

From this, we can see that

$$v^s_{\vec{p}} = \mathbf{C}\bar{u}^{sT}_{\vec{p}} \tag{10.50}$$

Exercise: check the following results:

$$\begin{aligned} &\bar{u}^s_{\vec{p}} u^{s'}_{\vec{p}} = 2m\delta^{ss'} \\ &\bar{v}^s_{\vec{p}} v^{s'}_{\vec{p}} = u^{sT}_{\vec{p}}\bar{\mathbf{C}}\mathbf{C}\,\bar{u}^{s'T}_{\vec{p}} = -2m\delta^{ss'} \qquad \text{with} \quad \bar{\mathbf{C}}\mathbf{C} = 1 \\ &\bar{u}^s_{\vec{p}}\gamma^0\, u^{s'}_{\vec{p}} = 2p^0\delta^{ss'} \\ &\bar{v}^s_{\vec{p}}\gamma^0\, u^{s'}_{\vec{p}} = 2p^0\delta^{ss'} \\ &\bar{v}u = 0 \\ &\bar{v}\gamma^0 u = 0 \end{aligned} \tag{10.51}$$

Remarks

- The notation $u^s(p)$ is often used instead of $u^s_{\vec{p}}$.
- We can define projectors to the positive and negative energy solutions:

$$\Lambda_+(p) = \frac{\not{p} + m}{2m} = \sum_{s=1,2} u^s_{\vec{p}} \otimes \bar{u}^s_{\vec{p}}$$
$$\Lambda_-(p) = \frac{-\not{p} + m}{2m} = \sum_{s=1,2} v^s_{\vec{p}} \otimes \bar{v}^s_{\vec{p}}$$

General solution

The most general solution to the Dirac equation is of course a linear combination of the two independent ones we have found so far:

$$\boxed{\Psi_\alpha(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_{s=1,2} \left\{ a^s(p) u^s_\alpha(p) e^{-ipx} + b^{s*}(p) v^s_\alpha(p) e^{ipx} \right\}} \quad (10.52)$$

with a Dirac index α. This is the analogue of the complex Φ solution of the Klein-Gordon equation.

10.6 Lagrangian formalism

Let us develop the Lagrangian formalism for the Dirac equation. We have the following Lagrangian and Lagrange density:

$$L = \int d^4x \mathcal{L}; \qquad \mathcal{L} = \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x) \quad (10.53)$$

Varying L with respect to Ψ and $\bar{\Psi}$ (like we varied the Klein-Gordon Lagrangian with respect to Φ and Φ^* before)

$$\begin{aligned} \frac{\delta L}{\delta \bar{\Psi}} &= \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = (i\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \\ \frac{\delta L}{\delta \Psi} &= \frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} = \bar{\Psi}(x)(-i\gamma^\mu \overleftarrow{\partial}_\mu - m) = 0 \end{aligned} \quad (10.54)$$

gives back the Dirac equation in two guises (the second one is the barred form). Notice that $\mathcal{L}$ is not symmetric in Ψ and $\bar{\Psi}$; one could distribute the derivative with respect to Ψ and $\bar{\Psi}$ symmetrically via partial integration in $S = \int d^4x \mathcal{L}$.

There is, of course, also a momentum field:

$$\boxed{\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_t \Psi_\alpha)} = i(\bar{\Psi}\gamma^0)_\alpha = i\Psi^\dagger_\alpha} \quad (10.55)$$

The barred equivalent gives

$$\frac{\partial \mathcal{L}}{\partial(\partial_t \bar{\Psi}_\alpha)} = \bar{\Pi}_\alpha = 0$$

if it does not take the symmetrized form mentioned above. This result ($\bar{\Pi}_\alpha = 0$) means that we have a *constrained system*, requiring special attention and special tools, like the Dirac bracket. There exists extensive literature about quantizing under constraints; here, we will not go into it.

The Hamiltonian density is given by

$$\begin{aligned}\mathcal{H} &= \Pi_\alpha \partial_t \Psi_\alpha - \mathcal{L} = i\bar{\Psi}\gamma^0\partial_0\Psi - \bar{\Psi}(x)(i\gamma^\mu p_\mu - m)\Psi(x) \\ &= i\bar{\Psi}\gamma^0\partial_0\Psi \end{aligned} \tag{10.56}$$

where in the last step we have applied the Dirac equation, which comes down to requiring the mass shell condition.

The Hamiltonian, after inserting eq. (10.52), finally takes the form

$$\boxed{H = \int d^3x \mathcal{H}(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} p^0 \sum_s \{a^{s*}(p)a^s(p) - b^s(p)b^{s*}(p)\}} \tag{10.57}$$

Exercise: check this (use eqs. (10.51)). Notice the minus sign in front of the b-term: this is different from the Klein-Gordon case, and will be of importance later on. Also note that the ordering of the a^s's and b^s's within their respective terms does not yet matter now, but will be important for the quantization procedure.

10.6.1 Quantization

We begin the quantization in the usual way, by promoting the functions $a^{s(*)}$ and $b^{s(*)}$ to operators $\mathbf{a}^{s(\dagger)}$ and $\mathbf{b}^{s(\dagger)}$, accompanied by the usual prescription that Poisson brackets are to be replaced with commutators. Now, we immediately run into trouble: this allows for limitlessly negative energy solutions , because of the minus sign in the Hamiltonian. To avoid this, we do not use commutators here, but take *anticommutators* instead:

$$\begin{aligned}\{\mathbf{a}^s(p), \mathbf{a}^{s'\dagger}(p')\} &= (2\pi)^3 2p^0 \delta^3(\vec{p} - \vec{p}')\delta^{ss'} \\ \{\mathbf{b}^s(p), \mathbf{b}^{s'\dagger}(p')\} &= (2\pi)^3 2p^0 \delta^3(\vec{p} - \vec{p}')\delta^{ss'} \\ \{\mathbf{a}, \mathbf{a}\} &= \{\mathbf{b}, \mathbf{b}\} = \{\mathbf{a}^\dagger, \mathbf{a}^\dagger\} = \{\mathbf{b}^\dagger, \mathbf{b}^\dagger\} \\ &= \{\mathbf{a}, \mathbf{b}\} = \{\mathbf{a}^\dagger, \mathbf{b}\} = \{\mathbf{a}, \mathbf{b}^\dagger\} = \{\mathbf{a}^\dagger, \mathbf{b}^\dagger\} = 0\end{aligned} \tag{10.58}$$

This is equivalent to

$$\boxed{\{\Psi_\alpha(\vec{x}, t), \bar{\Psi}_\beta(\vec{x}', t)\} = \delta^3(\vec{x} - \vec{x}')\gamma^0_{\alpha\beta}} \tag{10.59}$$

Of course, we could have obtained this directly, replacing the Poisson bracket of Ψ_α and $\Pi_\alpha = i\Psi^\dagger_\alpha = i(\bar\Psi\gamma^0)_\alpha$ by an anticommutator. Using this result, we get a more familiar-looking Hamiltonian:

$$H = \int \frac{d^3p}{(2\pi)^3 2p^0} p^0 \sum_s \Big\{ \mathbf{a}^{s\dagger}(p)\mathbf{a}^s(p) + \mathbf{b}^{s\dagger}(p)\mathbf{b}^s(p) - 1 \Big\} \tag{10.60}$$

from which we can conclude that the b-particles also have positive energy. This is one of the highlights of early QFT.

Fock space

We have already discussed this in the context of non-relativistic QM, in chapter 3. A general state $|\Psi\rangle$ is given by

$$\begin{aligned} |\Psi\rangle = \Big[\ c_0 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \sum_{s_1 \ldots s_n} \int \frac{d^3k_1}{(2\pi)^3 2k_1^0} \cdots \int \frac{d^3k_n}{(2\pi)^3 2k_n^0} \times \\ c_n(k_1, s_1, \ldots, k_n, s_n) \mathbf{a}^\dagger(k_1, s_1) \ldots \mathbf{a}^\dagger(k_n, s_n) \ \Big] |0\rangle + \\ \text{similar for } \mathbf{b}, \mathbf{b}^\dagger + \\ \text{mixed terms} \end{aligned} \tag{10.61}$$

The fact that $\{\mathbf{a}^\dagger, \mathbf{a}^\dagger\} = 0$ forces $\mathbf{a}^\dagger\mathbf{a}^\dagger = 0$, so there can never be more than one particle with momentum k and spin s; this is the Pauli principle. Due to the complete antisymmetry in the $\mathbf{a}^\dagger$'s, the coefficients c_n can be taken to be completely antisymmetric itself: any symmetric parts would vanish. Notice, by the way, that the ordering of the $\mathbf{a}$'s and $\mathbf{b}$'s is also important, due to the presence of the mixed terms.

10.6.2 Charge conjugation

We want to show that a-particles and b-particles have opposite charge. To this purpose, consider the coupling to the electromagnetic field, also called "minimal coupling":

$$\begin{aligned} \partial_\mu \to \mathcal{D}_\mu = \partial_\mu + ieA_\mu \qquad \text{so} \\ \mathcal{L} = \bar\Psi(x)(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\Psi(x) \end{aligned} \tag{10.62}$$

Let us now introduce the charge conjugation:

$$\boxed{\Psi(x) \xrightarrow{C} \Psi^C(x) = \mathbf{C}\bar\Psi^T(x)} \tag{10.63}$$

(This should correspond to exchanging Φ and Φ^* in the Klein-Gordon case.) $\mathbf{C}$ is given by

$$\mathbf{C} = i\gamma^0\gamma^2 = \bar{\mathbf{C}} \tag{10.64}$$

i.e., it is the one we have used in sec. 10.5, defined in eq. (10.47). The barred equivalent of eq. (10.63) is $\bar{\Psi}(x) \to \Psi^T \mathbf{C} = \bar{\Psi}^C$.

Let us further investigate $\mathbf{C}\bar{\Psi}^T(x)$:

$$\mathbf{C}\bar{\Psi}^T(x) = \mathbf{C}\gamma^{0T}\Psi^* =$$
$$\int \frac{d^3p}{(2\pi)^3 2p^0} \sum_s \left\{ \mathbf{a}^{s\dagger}(p)\mathbf{C}(\gamma^0)^T u^s(p) e^{ipx} + \mathbf{b}^{s\dagger}(p)\mathbf{C}(\gamma^0)^T v^s(p) e^{-ipx} \right\}$$

Using that $v^s(p) = \mathbf{C}\gamma^{\mu T} u^s(p)$ and $u^s(p) = \mathbf{C}\gamma^{\mu T} v^s(p)$, we see that this simply comes down to exchanging the roles of $a^s(p)$ and $b^s(p)$; in other words, *particles* and *antiparticles* are exchanged. Now consider the interaction term in the Lagrangian (eq. (10.62)): the Dirac equation becomes

$$(i\gamma^\mu(\partial_\mu + ieA_\mu - m)\Psi(x) = 0$$

Taking the "bar-conjugate" of the equation gives

$$\bar{\Psi}(-i\gamma^\mu(\overleftarrow{\partial}_\mu - ieA_\mu - m) = 0$$

Letting $\mathbf{C}$ act on this, we get

$$(-i\gamma^{\mu T}(\partial_\mu - ieA_\mu - m)\mathbf{C}^{-1}\mathbf{C}\bar{\Psi}^T = 0$$

And in the end, we see that in the final form,

$$(i\gamma^{\mu T}(\partial_\mu - ieA_\mu - m)\mathbf{C}\bar{\Psi}^T = 0$$

the sign of the charge has changed indeed.

Chapter 11

Feyman rules for theories with fermions

11.1 Bilinear Covariants

The Lagrange density is a Lorentz scalar, which has to be constructed to include fermions. The algebra of Dirac matrices has 16 elements, which are linearly independent 4x4 matrices:

$$\mathbb{1};\ \gamma^\mu;\ \sigma^\mu\nu = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right];\ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3;\ \gamma^5\gamma^\mu \tag{11.1}$$

(Check: there are $1 + 4 + 6 + 1 + 4 = 16$ of them and they are linearly independent.) Remember that a spinor transforms like

$$\Psi'_\alpha(x') = S_{\alpha\beta}(\Lambda)\Psi_\beta(x) \tag{11.2}$$

Its barred conjugate transforms like

$$\bar{\Psi}'_\alpha(x') = \bar{\Psi}_\beta(x)\bar{S}_{\beta\alpha} \qquad \text{with} \quad \bar{S} = \gamma^0 S^\dagger \gamma^0 \tag{11.3}$$

From $S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu\gamma^\mu$ it follows that $\bar{S} = S^{-1}$. Using this, we can see that $\bar{\Psi}\Psi$ is a Lorentz scalar:

$$\bar{\Psi}'_\alpha(x')\Psi'_\alpha(x') = \bar{\Psi}'_\alpha(x')\bar{S}S\Psi'_\alpha(x') = \bar{\Psi}_\beta(x)\Psi_\beta(x) \tag{11.4}$$

Similarly, we obtain

$$\bar{\Psi}'(x')\gamma^\nu\Psi'(x') = \bar{\Psi}\bar{S}\gamma^\nu S\Psi = \Lambda^\nu{}_\mu\bar{\Psi}(x)\gamma^\mu\Psi(x) \tag{11.5}$$

That is, $\bar{\Psi}(x)\gamma^\nu\Psi(x)$ is a Lorentz vector. More generally, if we denote the 16 Dirac matrices by $\Gamma^{(i)}$, then $\bar{\Psi}(x)\Gamma^{(i)}\Psi(x)$ is a tensor with the indices of the sandwiched Dirac matrix. Indeed, $\bar{\Psi}\sigma^{\mu\nu}\Psi$ is a symmetric, traceless tensor of rank 2.

With the transformation properties of γ^5 under proper Lorentz transformations and space inversion

$$\begin{aligned} S^{-1}(\Lambda)\gamma^5 S(\Lambda) &= \gamma^5 \\ S^{-1}(\Pi)\gamma^5 S(\Pi) &= -\gamma^5 \qquad (S(\Pi) = \gamma^0) \end{aligned}$$

(**exercise**: check this; remember that $\gamma^5 = \frac{i}{4!}\epsilon_{\lambda\mu\nu\sigma}\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\sigma$) we see that

- $\bar{\Psi}\gamma^5\Psi$ is a pseudoscalar
- $\bar{\Psi}\gamma^5\gamma^\mu\Psi$ is a pseudovector

This will be very important for constructing the coupling terms in the Lagrangian. For example, in the electromagnetic case we have

$$\mathcal{L}^{\text{elmg}} = \bar{\Psi}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\Psi \tag{11.6}$$

(minimal coupling) with as a possible additional coupling

$$\mathcal{L}^{\text{int}} = \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}$$

Another possibility is Yukawa coupling of a Dirac field to scalars and pseudoscalars, respectively. The latter one can, for example, be found in $\pi N\bar{N}$ coupling.

$$\mathcal{L}^{\text{sc}}_{\text{int}} = -f\bar{\Psi}(x)\Psi(x)\phi(x) \tag{11.7}$$

$$\mathcal{L}^{\text{ps.sc.}}_{\text{int}} = -f\bar{\Psi}(x)\gamma^5\Psi(x)\phi(x) \tag{11.8}$$

Note that in both, ϕ may be real or complex valued.

11.2 LSZ reduction

To calculate S-matrix elements with fermions one has to apply the LSZ reduction again. To this end, similar to the scalar case, one can use:

$$\begin{aligned} a^s_{\text{in}}(k) &= \int d^3x \bar{u}^s(k)\exp(ikx)\gamma^0\Psi_{\text{in}}(x) \\ b^s_{\text{in}}(k) &= \int d^3x \bar{v}^s(k)\exp(-ikx)\gamma^0\Psi_{\text{in}}(x) \end{aligned} \tag{11.9}$$

The reduction formula for n incoming particles ($u(k_i)$), n' incoming antiparticles ($\bar{v}(k'_i)$), m outgoing particles ($\bar{u}(q_i)$) and m' outgoing antiparticles ($v(q'_i)$) then reads

$$\begin{aligned} &\langle 0|\, b_{\text{out}}(q'_1)\ldots a_{\text{out}}(q_1)\ldots a^\dagger_{\text{in}}(k_1)\ldots b^\dagger_{\text{in}}(k'_1)\ldots|0\rangle = \\ &\quad (-iZ_2^{-1/2})^{(n+m)}(-iZ_2^{-1/2})^{(n'+m')}\int d^4x_1\ldots d^4y'_1 \\ &\quad \exp\Big\{-\sum(k\cdot x + k'\cdot x' - q\cdot y - q'\cdot y')\Big\}\,\bar{u}(q_1)(i\partial\!\!\!/_{y_1} - m)\ldots\bar{v}(k_1)(i\partial\!\!\!/_{x'_1} - m) \\ &\quad \langle 0|\,\mathrm{T}\left\{\bar{\Psi}(y'_1)\ldots\Psi(y_1)\bar{\Psi}(x_1)\ldots\Psi(x'_1)\right\}|0\rangle\,(-i\overleftarrow{\partial\!\!\!/}_{x_1} - m)u(k_1)\ldots(-i\overleftarrow{\partial\!\!\!/}_{y'_1} - m)v(q'_1) \end{aligned} \tag{11.10}$$

where the disconnected part has been left out like before. Dirac and spin indices have been suppressed in this formula, to improve the legibility. **Be careful**: anticommutations in the T-product will produce minus signs!

Remark

For photons, we have

$$\langle \beta; k, \epsilon\, out | \alpha_{in} \rangle = (-i)(Z_3)^{-1/2} \int d^4x \exp(ikx) \langle \beta_{\text{out}} | \epsilon j(x) | \alpha_{\text{in}} \rangle \quad (11.11)$$

$$\text{with} \quad \partial^2 A^\mu = j^\mu \quad (11.12)$$

Here, further reduction requires a redefinition of the T-product (called T*-product), adding distributions ($\neq 0$ for $x = y$) in order to obtain covariant expressions.

11.3 Feynman rules

Here we can be rather short:

(i) One derives the Gell-Mann-Low formula again, this time formulated in Ψ^{in} and $\bar{\Psi}^{\text{in}}$ fields.

(ii) The analogous Wick theorem for fermionic fields leads to "contractions" of the spinor fields. Since the Dirac theory is similar to that of complex scalars, it is pretty obvious (**exercise**: check this) that only the propagator

$$\langle 0 | T \left(\Psi_\alpha(x) \bar{\Psi}_\beta(y) \right) | 0 \rangle$$

is not zero

Note

Fermions can also be treated in the path integral formalism very elegantly. We will come back to this in chapter 13.

11.3.1 The Dirac propagator

Let us start off by defining

$$S_{\text{F},\alpha\beta}(x - y) := \langle 0 | T \left(\Psi_\alpha(x) \bar{\Psi}_\beta(y) \right) | 0 \rangle \quad (11.13)$$

where α and β are spinor indices. It is often notated as $S_{\text{F}}(x, y)$, and fulfills the equation

$$\boxed{\left((i\partial\!\!\!/ - m)_x \right)_{\alpha\beta} \left(S_{\text{F}}(x, y) \right)_{\beta\gamma} = i\delta_{\alpha\gamma}\delta^4(x - y)} \quad (11.14)$$

We can check that $S_F(x-y) = (i\partial\!\!\!/ + m)_x D_F(x-y)$ is the solution with Feynman boundary conditions using

$$(i\partial\!\!\!/ - m)\,(i\partial\!\!\!/ + m) = -\partial^2 - m^2 = -(\partial^2 + m^2) \tag{11.15}$$

The Fourier transform turns out to be

$$S_F(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} \tag{11.16}$$

Of course, this propagator has direction, and the corresponding propagator line has an arrow, since Ψ is a complex field.

Note

In the time ordering of the LSZ reduction formula for Dirac fermions commutations *cause minus signs*. Similarly, in the Wick formula, contraction can only be done after commuting through the field in between and thereby picking up minus signs. There are $u, \bar{u}, v, \bar{v}$ in the LSZ-formula corresponing to ingoing and outgoing particles and ingoing and outgoing antiparticles. The vacuum graphs again cancel. Inserting the vacuum expectation value into the LSZ formula the outer propagators are cancelled, which is also referred to as "truncated", and we are left with a $\sqrt{Z}$ factor. This will be important for loop calculations later on.

Later, we will see that in the dressed propagator for fermions, the "self-energy" insertions also have the Dirac structure:

$$\frac{i(p\!\!\!/ + m_0)}{p^2 - m^2 + i\epsilon} + S_{\mathrm{F}}(p, m_0)\Pi S_{\mathrm{F}}(p, m_0) + \cdots = \frac{iZ_2(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} + \text{rest}$$

We then obtain the Feynman rules for S-matrix elements (in momentum space):

For Yukawa theory

(i) Dirac propagator (eq. (11.16), instead of the scalar propagator before)

(ii) $-if$ or $-if\gamma_5$, for scalar and pseudoscalar vertices, respectively

(iii) statistical factors

(iv) vertex integration

(v) u^s for incoming particles, $\bar{u}^s$ for outgoing particles, v^s for outgoing antiparticles, $\bar{v}^s$ for incoming antiparticles

(vi) $\sqrt{Z}$ for outer scalar, $\sqrt{Z_2}$ for outer fermion fields

(vii) -1 for closed (inner) fermion lines (to be explained later)

Note: fermion propagator lines end in outer particles or are closed, but do not cross each other.

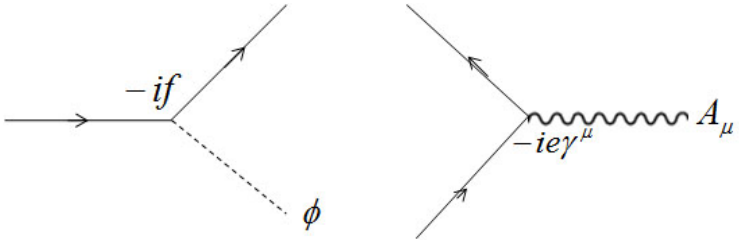


Figure 11.1: Yukawa scalar and QED vertices

For quantum electrodynamics

The rules will just be given here, in the Lorentz gauge ($\partial^\mu A_\mu = 0$). The derivation will come later, in the context of the fermionic path integral.

The interaction Hamiltonian is

$$\mathbf{H}_{\text{int}} = -eQ\bar{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x) \tag{11.17}$$

(Q is the charge number, e.g. -1 for electrons). The rules are:

(i) photon propagator: $\frac{-ig_{\mu\nu}}{p^2+i\epsilon}$ (the '$-m$'-term in the denominator disappears, since $m = 0$ for photons); Dirac propagator as before

(ii) $ie\gamma^\mu Q$ for vertices

(iii) statistical factors

(iv) vertex integration

(v) u^s, $\bar{u}^s$, v^s, $\bar{v}^s$ as before

(vi) $\sqrt{Z_3}$ for outer photons

(vii) -1 for fermion loops

Note

Spin averaging over incoming particles and spin *summation* over outgoing particles are often required when calculating cross sections. This gives, for example,

$$\frac{1}{2}\sum_{s,s'}\bar{u}^s(p_2)\gamma_5 u^{s'}(q_2)(\bar{u}^s(p_2)\gamma_5 u^{s'}(q_2))^* =$$
$$\frac{1}{2}\sum_{s,s'}\bar{u}^s(p_2)\gamma_5 u^{s'}(q_2)\bar{u}^{s'}(q_2)\gamma_5 u^s(p_2) =$$
$$\frac{1}{2}\sum_{s}\mathrm{tr}\left(u^s(p_2)u^s(p_2)\gamma_5\frac{\not{q}_2+m}{2m}\gamma_5\right) =$$
$$\frac{1}{2}\mathrm{tr}\left(\frac{\not{p}_2+m}{2m}\gamma_5\frac{\not{q}_2+m}{2m}\gamma_5\right) =$$
$$\frac{1}{2}\mathrm{tr}((\not{p}_2+m)(-\not{q}_2+m)) = 2(-p_2\cdot q_2+m^2)$$

(the factor of $\frac{1}{2}$ comes from spin averaging).

11.4 Simple example in Yukawa theory

Let us consider "nucleon scattering" with pion-exchange as force mediator as an example (between quotes because, of course, we know nowadays that nucleons are composite objects). The interaction Hamiltonian density is given by

$$\mathcal{H}_{\mathrm{int}} = f\bar{\Psi}\gamma_5\Psi\Phi \tag{11.18}$$

N-N scattering

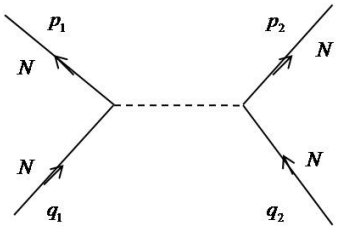


Figure 11.2: t-channel exchange

This is *t-channel exchange*: $t = (p_1 - q_1)^2$. The expression corresponding to the diagram (see figure) is the following:

$$(-if)^2\bar{u}(p_1)\,\gamma_5 u(q_1)\bar{u}(p_2)\gamma_5 u(q_2)\frac{i}{(p_1-q_1)^2-m^2+i\epsilon}\times$$
$$(Z_2)^2(2\pi)^4\delta^4(p_1+p_2-q_1-q_2) =$$
$$i\mathbf{M}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2) \tag{11.19}$$

There is a second diagram for N-N scattering, where the two outgoing lines are exchanged. According to the above recipe, this will give a minus sign; this can also be derived from a careful evaluation of the LSZ-formula.

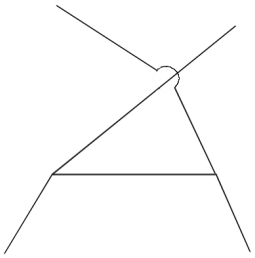


Figure 11.3: u-channel exchange

The Z-factors are not of great interest here; they will become important in the discussion of renormalization. The propagator $1/(p_1 - q_1)^2 - m^2$ in the t-channel gives, after Fourier transformation of the evaluated $i\mathbf{M}$ (the amplitude), the Yukawa potential, which is well known in nuclear physics.

N-$\bar{\text{N}}$ scattering

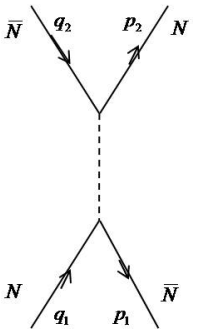


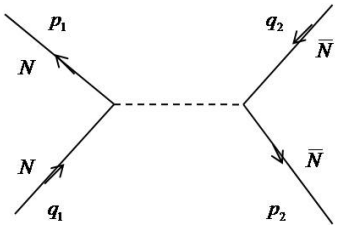
Figure 11.4: s-channel exchange

This is s-channel exchange: $s = (q_1 + p_2)^2$. The diagram is the same, except for the exchange of p_1 and q_2. The corresponding expression is

$$(-if)^2 \bar{v}(-p_1)\,\gamma_5 u(q_1)\bar{u}(p_2)\gamma_5 v(-q_2)\frac{i}{(p_1 - q_1)^2 + m^2 + i\epsilon}\times \\ (Z_2)^2(2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2) \qquad (11.20)$$

Another diagram that represents this type of scattering is shown below, and has the following expression:

$$(-if)^2 \bar{u}(p_1)\gamma_5 u(q_1)\bar{v}(-p_2)\gamma_5 v(-q_2)\frac{i}{(p_1-q_1)^2+m^2+i\epsilon}\times\ldots \qquad (11.21)$$



Notes

- In Peskin & Schröder, in the spirit of time-ordered perturbation theory, outer one-particle states are directly contracted with vertex-fields:

$$\langle p_1 q_2|\,\bar{\Psi}\gamma_5\Psi\bar{\Psi}\gamma_5\Psi\,|q_1 p_2\rangle$$

  The required permutations give rise to extra minus signs, which include signs related to the fermion statistics of the outer states. Altogether, this gives the $\bar{u}, u, \bar{v}$ and v as above.

- A closed fermion loop gives a minus sign, as mentioned above:

$$\langle 0|\,\mathrm{T}(\bar{\Psi}(x_1)\Psi(x_1)\bar{\Psi}(x_2)\Psi(x_2))\,|0\rangle \rightarrow$$
$$-\,\underbrace{\Psi(x_1)\bar{\Psi}}(x_2)\underbrace{\Psi(x_2)\bar{\Psi}}(x_1)$$

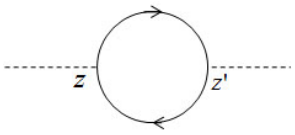


Figure 11.5: Closed fermion loop

Feynman approach vs. time-ordered perturbation theory

In the Feynman approach the propagators are off-shell and one has 4-momentum conservation at the vertices.

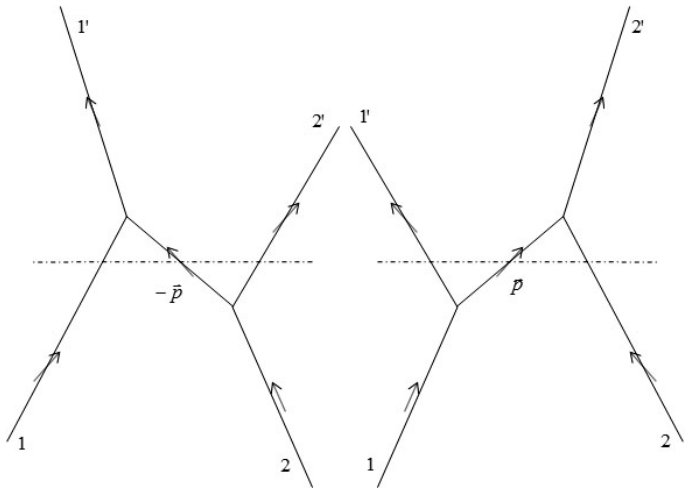


Figure 11.6: Cases (α) (left) and (β) (right)

In *time-ordered perturbation theory*, on the other hand, one has the well-known energy denominators, like in QM, and, which is new in field theory, 3-momentum conservation at the vertices. Of course the incoming and outgoing particles have the same total energy, but in between, energy is not conserved, as a consequence of the energy-time uncertainty relation. The intermediate particles are *on shell* in this approach.

For tree-level diagrams, the relation between the two approaches can be seen by decomposing the propagator:

$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{p_0^2 - \vec{p}^2 - m^2 + i\epsilon}$$

In the Feynman case, one rewrites this as

$$\left(\frac{1}{p_0 - \sqrt{\vec{p}^2 + m^2} + i\epsilon} - \frac{1}{p_0 + \sqrt{\vec{p}^2 + m^2} - i\epsilon} \right) \frac{1}{2p_{0,+}}$$

In the time-ordered perturbation theory, one has the following denominators:

$$E_1' + \sqrt{\vec{p}^2 + m^2} + E_2 - E_1 - E_2 = -p_0 + \sqrt{\vec{p}^2 + m^2} \qquad (\alpha)$$
$$E_1 + \sqrt{\vec{p}^2 + m^2} + E_2' - E_1' - E_2' = p_0 + \sqrt{\vec{p}^2 + m^2} \qquad (\beta)$$

These are the same as the ones from the Feynman case.

For loop integrals, the Feynman approach has

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon}$$

Time-ordered perturbation theory has

$$\int d^3k \frac{1}{\text{energy denominators}}$$

Here, the connection between the two approaches can be seen by doing the dk^0-integration using Cauchy's theorem (the residue theorem), closing the integration contour in one of the two halfplanes. One gets poles in k_0, yielding the required energy denominator in the residue.

For example, in Φ^3 scalar theory, for the graph in figure 11.6, one has

$$(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p-k)^2 - m^2 + i\epsilon} =$$
$$(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^0 - \sqrt{\vec{k}^2 + m^2} + i\epsilon} - \frac{1}{k^0 + \sqrt{\vec{k}^2 + m^2} - i\epsilon} \right) \frac{1}{2k^{0,+}} \times$$
$$\left(\frac{1}{(p^0 - k^0) - \sqrt{(\vec{p} - \vec{k})^2 + m^2} + i\epsilon} - \frac{1}{(p^0 - k^0) + \sqrt{(\vec{p} - \vec{k})^2 + m^2} - i\epsilon} \right) \times$$
$$\frac{1}{2(p^0 - k^0)^+}$$

(This is a rough calculation, just to see the principle; we will not worry about infinities here.) Closing the integration contour in the lower half plane and applying the residue theorem gives

$$(i\lambda)^2 \int \frac{d^3k}{(2\pi)^4} (-2\pi i) \frac{1}{2k^{0,+}} \frac{1}{2(p^0 - k^0)^+} \times$$
$$\left(\frac{1}{p^0 - \sqrt{\vec{k} + m^2} - \sqrt{(\vec{p} - \vec{k})^2 + m^2}} - \frac{1}{p^0 + \sqrt{\vec{k} + m^2} + \sqrt{(\vec{p} - \vec{k})^2 + m^2}} \right.$$
$$\left. - \frac{1}{p^0 - \sqrt{\vec{k} + m^2} + \sqrt{(\vec{p} - \vec{k})^2 + m^2}} + \frac{1}{p^0 - \sqrt{\vec{k} + m^2} + \sqrt{(\vec{p} - \vec{k})^2 + m^2}} \right)$$

Chapter 12

Quantum mechanical interpretation of the Dirac equation

In this chapter, we will investigate the relation between the Dirac and Schrödinger equations in the non-relativistic limit, and study the Foldy-Wouthousen transformation.

12.1 Interpretation as Schrödinger equation

If one reinserts the β- and α-matrices used in deriving the Dirac equation (chapter 10), it takes the form

$$i\partial_t \Psi(\vec{x},t) = \underbrace{\left(m\beta + \frac{\vec{\alpha}\cdot\vec{\nabla}}{i}\right)}_{\mathbf{H}} \Psi(\vec{x},t) \tag{12.1}$$

This looks very similar to the Schrödinger equation. It is Hermitean ($\beta = \beta^\dagger$ and $\vec{\alpha} = \vec{\alpha}^\dagger$) and $\mathbf{H}^2 = m^2 + \vec{p}^2$. Defining ρ by

$$\rho(\vec{x},t) = \Psi^\dagger\Psi = \bar{\Psi}\gamma^0\Psi \tag{12.2}$$

one obtains a continuity equation:

$$\partial_\mu(\bar{\Psi}\gamma^\mu\Psi) = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\partial_\mu\gamma^\mu\Psi = 0 \tag{12.3}$$

where in the last step, we have used the Dirac equation: $(i\gamma^\mu\partial_\mu - m)\Psi = 0$, and its barred couterpart $\partial_\mu\bar{\Psi}(-i\bar{\gamma}^\mu) - \bar{\Psi}m = 0$,

All in all, it looks like we could try to interpret the Dirac equation like a Schrödinger equation without "second quantization". However, there is one important difference: the Dirac equation has negative energy solutions. The

Hamiltonian above is not bounded from below in any way; we would have to put this in artifically, since the Schrödinger equation has only positive energy solutions. This means we would assume

$$\Psi(x) = \int \frac{d^3p}{2p^0(2\pi)^3} \sum_s a^s(p) u^s(p) e^{-ipx} \tag{12.4}$$

without the b-term we had in chapter 10. If we do this,

$$\int d^3x \Psi^\dagger \Psi = \int d^3x \bar{\Psi}\gamma^0\Psi = \int d^3p \widetilde{\Psi}^\dagger(p)\widetilde{\Psi}(p) \tag{12.5}$$

has a probability interpretation. Note that

$$\widetilde{\Psi}(p) = \sum_s \frac{a^s(p)u^s(p)}{((2\pi)^3 2p_0)^{1/2}} \tag{12.6}$$

We want to be able to calculate expectation values

$$\langle\Psi|\, A\, |\Psi\rangle = \int d^3x \Psi^\dagger A \Psi$$

like in QM. Assuming an $\vec{\mathbf{X}}$ which acts on $\Psi(x)$ like a multiplication by $\vec{x}$, we can calculate, by Ehrenfest's theorem,

$$\frac{d}{dt}\int d^3x \Psi^\dagger \vec{\mathbf{X}} \Psi = i \int d^3x \Psi^\dagger [\mathbf{H}, \vec{\mathbf{X}}]\Psi$$

Writing out the commutator in the integrand, we have

$$\begin{aligned} [-i\vec{\alpha}\cdot\vec{\nabla} + \beta m, \vec{\mathbf{X}}] &= -i\vec{\alpha} \quad \text{so} \\ \frac{d}{dt}\langle\vec{x}\rangle &= \langle\vec{\alpha}\rangle \end{aligned} \tag{12.7}$$

i.e., α acts like a velocity. Since

$$[\mathbf{H}, \vec{\alpha}] \neq 0$$

we know that this "velocity" changes. Its expectation value, assuming both positive and negative energy solutions, is

$$\begin{aligned} \langle\vec{\alpha}\rangle = &\int \frac{d^3p}{(2\pi)^3 2p_0}\frac{\vec{p}}{p_0}\sum_s \left\{|a^s|^2 + |b^s|^2\right\} + \\ &\int \frac{d^3p}{(2\pi)^3 2p_0}\frac{1}{2p_0}\sum_{s,s'=1}^{2}\Big\{a^{s'*}(-\vec{p})b^{s'*}(\vec{p})\bar{u}^{s'}(\vec{p})\vec{\alpha}v^s(\vec{p})e^{2pi_0x^0} + \\ &b^{s'}(-\vec{p})a^s(\vec{p})\bar{v}^{s'}(\vec{p})\vec{\alpha}u^s(-\vec{p})e^{-2ip_0x^0}\Big\} \end{aligned} \tag{12.8}$$

The first term reproduces the usual group velocity $\vec{p}/p_0$; the last term mixes the positive and negative energy solutions (more properly said, it mixes the a- and b-modes; the energies p_0 are always positive). If $\vec{\alpha}$ is a velocity, and $\alpha_i^2 = 1$ as we have seen before, this must be interpreted as a *zitterbewegung*: a movement with the velocity of light and changing direction. This movement arises since operators corresponding to interactions, e.g. $\vec{\mathbf{X}}$, lead out of the space of positive frequencies. This leads to the question why radiative transitions from positive to negative energies are impossible.

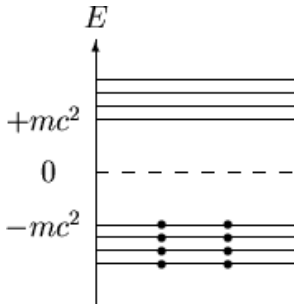


Figure 12.1: The Dirac sea, with occupied negative energy states

Here, we will introduce the concept of the *Dirac sea* (fig. 12.1[1]). Dirac postulated that all states with negative energy are occupied in the ground state of the system. In this case, the Pauli principle forbids transition to these states, which together form the Dirac sea. In this view, the excitation of an electron with negative energy, $E_e < -m_e$, due to electromagnetic radiation with $E_\gamma > 2m_e$ results in an electron with positive energy and a *hole*, i.e. an unoccupied negative energy level. This hole behaves like a particle with opposite charge, i.e. a positron, and therefore, the above process is just that of pair creation. Pair annihilation occurs when an electron emits electromagnetic radiation and jumps down into an existing hole:

$$\boxed{e^+ + e^- \to 2\gamma} \tag{12.9}$$

The fact that there are two photons being produced is a consequence of certain conservation laws we will return to at a later time. The relation between positive and negative energy solutions is given by a kind of charge conjugation. Be careful, however: in this picture, there are no anticommuting Fourier coefficients. It may at times pop up in old literature or in solid state physics, but we will use the "second quantized picture".

[1] Taken from http://www.phys.ualberta.ca/ gingrich/phys512/latex2html/node63.html, accessed 30-09-2007.

12.2 Non-relativistic limit

12.2.1 Pauli equation

In the non-relativistic limit, the Dirac-Pauli choice of γ-matrices is convenient. A short reminder:

$$
\begin{aligned}
{}^{\mathrm{D}}\gamma^0 = \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad {}^{\mathrm{D}}\gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\
{}^{\mathrm{D}}u_p^s = S u_p^{s\,\mathrm{W}} = \frac{1}{(p_0+m)^{1/2}} \begin{pmatrix} (p_0+m)\chi^s \\ \vec{\sigma}\cdot\vec{p}\chi^s \end{pmatrix}
\end{aligned} \tag{12.10}
$$

Note that this u is a factor $m^{1/2}$ smaller than the last time we saw it (chapter 10); this is done to ensure that $u^\dagger u = 1$, to preserve the analogy to QM.

In the non-relativistic limit,

$$
p_0 = m + \frac{1}{2}\frac{\vec{p}^2}{2m} + \ldots \qquad \text{for } |\vec{p}| \ll m \tag{12.11}
$$

In this limit, the upper components of u are much bigger than the lower ones. This distinction between big and small components extends to the covariant quantities:

$$
\begin{aligned}
&\bar\Psi\Psi = \Psi^\dagger\gamma^0\Psi && \text{and} \quad \bar\Psi\gamma^0\Psi = \Psi^\dagger \overbrace{(\gamma^0)^2}^{=\mathbb{1}}\Psi && \text{are big} \\
&\bar\Psi\gamma^5\Psi = \Psi^\dagger\gamma^0\gamma^5\Psi && \text{and} \quad \bar\Psi\gamma^i\Psi = \Psi^\dagger\gamma^0\gamma^i\Psi && \text{are small} \\
&\bar\Psi\sigma^{0i}\Psi = \Psi^\dagger \overbrace{\gamma^0\gamma^0}^{=\mathbb{1}}\gamma^i\Psi && \text{is small} \\
&\bar\Psi\sigma^{ik}\Psi = \Psi^\dagger\gamma^0\gamma^i\gamma^k\Psi && \text{is big}
\end{aligned}
$$

As mentioned before, we write the Dirac equation as

$$
i\partial_t\Psi(\vec{x},t) = \left(m\beta + \frac{\vec{\alpha}\cdot\vec{\nabla}}{i}\right)\Psi(\vec{x},t)
$$

We will call the "Hamiltonian" $\mathbf{H}_{\mathrm{D}}$:

$$
\left(m\beta + \frac{\vec{\alpha}\cdot\vec{\nabla}}{i}\right) = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{\nabla}/i \\ \vec{\sigma}\cdot\vec{\nabla}/i & -m \end{pmatrix} = \mathbf{H}_{\mathrm{D}} \tag{12.12}
$$

Introducing electromagnetic interaction, we replace ∂_μ by $\mathcal{D}_\mu$:

$$
\partial_\mu \to \partial_\mu + ieA_\mu = \mathcal{D}_\mu
$$

With the ansatz

$$
\Psi = e^{-imx^0}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{12.13}
$$

we obtain two coupled equations:

$$i\partial_0\varphi_1 = -i\vec{\sigma}\cdot(\vec{\nabla}+ie\vec{A})\varphi_2 - e\phi\varphi_1 \tag{12.14}$$
$$i\partial_0\varphi_2 = -i\vec{\sigma}\cdot(\vec{\nabla}+ie\vec{A})\varphi_1 - e\phi\varphi_2 - 2m\varphi_2$$

We want stationary solutions with positive energy $\tilde{E} = E - m \ll m$ and $\varphi \approx e^{-i\tilde{E}x^0}$. This condition implies

$$|i\partial_0\varphi_2| \ll 2m\varphi_2 \qquad \text{and} \tag{12.15}$$
$$|e\phi| \ll m \qquad \text{(weak potential)} \tag{12.16}$$

This gives for the small components φ_2:

$$\boxed{\varphi_2 \approx -i\vec{\sigma}\cdot\frac{(\vec{\nabla}+ie\vec{A})}{2m}\varphi_1} \tag{12.17}$$

Inserting this into the equation for φ_1 gives

$$i\partial_0\varphi_1 \approx \left[\frac{-\vec{\sigma}(\vec{\nabla}+ie\vec{A})\vec{\sigma}\cdot(\vec{\nabla}+ie\vec{A})}{2m} - e\phi\right]\varphi_1$$

For the vector product between the brackets, we have the following identity:

$$\vec{\sigma}\cdot\vec{a}\vec{\sigma}\vec{b} = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

(**Exercise**: check this.) Plugging this in gives

$$i\partial_0\varphi_1 \approx \left[-\frac{(\vec{\nabla}+ie\vec{A})^2}{2m} + \frac{e}{2m}\vec{\sigma}\cdot\vec{B} - e\phi\right]\varphi_1$$

with

$$\vec{A} = \frac{\vec{B}\times\vec{r}}{2} \tag{12.18}$$

and constant $\vec{B}$. This is the coordinate gauge, also known as the *Fock-Schwinger gauge*. In this simple case, writing out the square between the brackets and rearranging terms leads us to the Pauli-equation:

$$\boxed{i\partial_0\varphi_1 \approx \left[-\frac{\vec{\nabla}^2}{2m} + \frac{e}{2m}\left(\vec{L} + 2\frac{\vec{\sigma}}{2}\right)\cdot\vec{B} - e\phi\right]\varphi_1} \tag{12.19}$$

The factor 2 in front of $\vec{\sigma}/2$ should actually be g, the magnetic dipole moment. In the Pauli equation, which therefore is an approximation, it is assumed to be 2.

Remark

The Fock-Schwinger gauge can also be formulated as follows:

$$x^\mu A_\mu(x) = 0$$
$$A_\mu(x) = \int_0^1 d\eta\, \eta x^\nu F_{\nu\mu}(\eta x)$$

12.2.2 Foldy-Wouthousen transformation

It would be useful to diagonalize $\mathbf{H}_\mathrm{D}$, in order to have the positive energy solution only in the upper components. This can be done by the *Foldy-Wouthousen transformation*:

$$\mathbf{H}'_\mathrm{FW} = \mathbf{U}\mathbf{H}_\mathrm{D}\mathbf{U}^\dagger \tag{12.20}$$

Free Dirac equation

For the free Dirac equation, $\mathbf{U}$ is given by

$$\boxed{\mathbf{U} = \left(\frac{p_0}{2(p_0+m)}\right)^{1/2} \frac{p_0 + m - \vec{\gamma}\cdot\vec{p}}{p_0}} \tag{12.21}$$

Note that with $\gamma^{0\dagger} = \gamma^0$ and $\vec{\gamma}^\dagger = -\vec{\gamma}$, this means that $\mathbf{U}^\dagger\mathbf{U} = 1$. One can also rewrite the numerator of the second fraction as $-(\gamma p - m) + \gamma^0 p_0 + p_0$; Björken & Drell, and Itzykson & Zuber use yet another notation:

$$\mathbf{U} = e^{iS} \qquad \text{with} \quad S = -i\vec{\gamma}\frac{\vec{p}}{|\vec{p}|}\theta(p) \tag{12.22}$$
$$\text{and} \quad \tan(2\theta(p)) = \frac{|\vec{p}|}{m}$$

The action of $\mathbf{U}$ on a spinor u is as follows:

$$\mathbf{U}u^s(p) = \left(\frac{p_0}{2(p_0+m)}\right)^{1/2} (1+\gamma^0)u^s(p) \tag{12.23}$$

Here, we have used the Dirac equation to get rid of the term with $(\gamma p - m)$. The factor $(1+\gamma^0)$ selects the upper components, like $(1+\gamma^5)$ in the Weyl representation (see chapter 10). The Hamiltonian itself, finally, takes the form

$$\mathbf{H}'_\mathrm{FW} = \mathbf{U}\mathbf{H}\mathbf{U}^\dagger = \frac{p_0}{2(p_0+m)}\frac{1}{p_0^2}(-\vec{\gamma}\cdot\vec{p} + m + p_0)(-\gamma^0\vec{\gamma}\cdot\vec{p} + m\gamma^0)\times \tag{12.24}$$
$$(\vec{\gamma}\cdot\vec{p} + m + p_0) = \gamma^0 p_0 = \gamma^0(m^2 + \vec{p}^2)^{1/2}$$

Note that γ^0 is diagonal in the Dirac-Pauli basis, so this Hamiltonian is diagonal as well, which was the original aim of the transformation.

Remark

The action of $\vec{\mathbf{X}}_{\mathrm{FW}}$ in x-space should be multiplication with $\vec{x}$. This implies that it must be $i\vec{\nabla}_p$; hence,

$$\vec{\mathbf{X}}_{\mathrm{D}} = \mathbf{U}^{\dagger}\vec{\mathbf{X}}_{\mathrm{FW}}\mathbf{U}$$

must be given by

$$\vec{\mathbf{X}}_{\mathrm{D}} = i\vec{\nabla}_p - \frac{i\vec{\gamma}}{2p_0} + i\frac{\vec{\gamma}\cdot\vec{p}\frac{\vec{p}}{p_0} + i[\vec{\Sigma}\times\vec{p}]}{2p_0(p_0+m)}$$

with

$$\Sigma^l = \varepsilon^{lik}\sigma^{ik} = \frac{i}{2}(\gamma^i\gamma^k - \gamma^k\gamma^i)\varepsilon^{lik}$$

Exercise: check this.

With this $\vec{\mathbf{X}}_{\mathrm{D}}$, we have the following commutation relations:

$$\begin{aligned} [X^i_{\mathrm{D}}, X^k_{\mathrm{D}}] &= 0 \\ [X^i_{\mathrm{D}}, P^j] &= i\delta^{ij} \qquad \text{N.b.: } \vec{P}_{\mathrm{D}} = \vec{P}_{\mathrm{FW}} \\ [\mathbf{H}'_{\mathrm{FW}}, X_{\mathrm{FW}}] &= i\gamma^0\frac{\vec{p}}{p_0} \end{aligned} \tag{12.25}$$

The above relations imply

$$\frac{d}{dt}\left\langle\vec{\mathbf{X}}_{\mathrm{D}}\right\rangle = i\left\langle[\mathbf{H}_{\mathrm{D}}, \vec{\mathbf{X}}_{\mathrm{D}}]\right\rangle = i\left\langle[\mathbf{H}'_{\mathrm{FW}}, \vec{\mathbf{X}}_{\mathrm{FW}}]\right\rangle = \left\langle\frac{\vec{p}}{p_0}\right\rangle \tag{12.26}$$

in accordance with Ehrenfest's theorem. With this $\vec{\mathbf{X}}_{\mathrm{D}}$, $\vec{\mathbf{X}}_{\mathrm{D}}\Psi^{(+)}$ contains only positive energy solutions. The above means that we can treat problems in the Foldy-Wouthousen representation and then transform to the Dirac representation by means of $\mathbf{U}$.

12.2.3 Foldy-Wouthousen representation in the presence of a constant electromagnetic field

To evaluate the Dirac equation in the presence of a constant magnetic field, we will use the Foldy-Wouthousen representation in the notation mentioned in eq. (12.22). We expand the exponential:

$$\tilde{\mathbf{H}}_{\mathrm{FW}} = e^{iS}\mathbf{H}_{\mathrm{D}}e^{-iS} = \mathbf{H}_{\mathrm{D}} + i[S,\mathbf{H}_{\mathrm{D}}] + \frac{i^2}{2!}[S,[S,\mathbf{H}_{\mathrm{D}}]] + \frac{i^3}{3!}[S,[S,[S,\mathbf{H}_{\mathrm{D}}]]]$$

We want to evaluate the equation up to order $\mathcal{O}(m^{-2})$. Observe that $S \propto m^{-1}$ and $\mathbf{H}_{\mathrm{D}} = \beta m + e\phi + \vec{\alpha}\vec{p} \approx m$, since the mass is much larger than the momentum. With the above equation in mind, we can make the following ansatz:

$$\tilde{\mathbf{H}}_{\mathrm{FW}} = m\gamma^0 + h^{(0)} + \frac{h^{(1)}}{m} + \dots \tag{12.27}$$

Let us decompose S:

$$e^{iS} = e^{iS_2} e^{iS_1}$$

into the following S_1 and S_2:

$$\begin{aligned} S_1 &= \frac{-i}{2m}\vec{\gamma}(\vec{p} - e\vec{A}) \\ S_2 &= \frac{-i}{m^2}\left(\frac{1}{4}ie\vec{\alpha}\cdot\vec{E} - \frac{\gamma^0}{6m}(\vec{\alpha}\cdot(\vec{p} - e\vec{A}))^3\right) \end{aligned}$$

Since $[S_1, S_2] \propto m^{-3}$, we don't have to consider these terms. This gives for $\mathbf{H}^{(2)}$:

$$\mathbf{H}^{(2)} = m + e\phi + \frac{1}{2m}\left(\frac{\vec{\nabla}}{i} - e\vec{A}\right)^2 - \frac{e}{2m}\vec{\sigma}\cdot\vec{B} - \frac{e}{gm^2}8\pi\rho + \tag{12.28}$$

$$\frac{e}{4m^2}\frac{1}{r}\frac{\partial\phi}{\partial r}(\vec{\sigma}\cdot\vec{L}) + \mathcal{O}(m^{-3}) \tag{12.29}$$

The sixth term represents the *spin-orbit coupling*, with the Thomas factor. The fifth term (the last one on the first line) is known as the *Darwin term*. ρ is the charge density, given by $\rho = -e\delta^3(\vec{x})$. The electron experiences a potential $V(\vec{r} + \delta\vec{r})$ because of the *zitterbewegung*. Varying V and Taylor expanding ΔV gives roughly this term:

$$\overline{\Delta V} \approx \frac{1}{2}\cdot\frac{1}{3}\frac{1}{m^2}\vec{\nabla}^2 V = \frac{e}{6}\frac{1}{m^2}\delta^3(\vec{x})$$

The Foldy-Wouthousen representation is fine for concrete non-relativistic calculations. It will not do for fundamental questions concerning interactions, in particular since minimal coupling does not hold. That is, it works only for fundamental particles like e, μ, q, etc., but not for protons or neutrons.

12.3 The hydrogen atom

For the hydrogen atom, we have

$$\mathbf{H} = \frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + \beta m - \frac{e^2}{r} \quad \text{and} \quad \Psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{12.30}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \text{and} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

Plugging all of this into the stationary Dirac equation gives

$$\boxed{\begin{array}{l}\left(m-\frac{e^2}{r}\right)\phi_1 - i\vec{\sigma}\cdot\vec{\nabla}\phi_2 = E\phi_1 \\ \left(-m-\frac{e^2}{r}\right)\phi_2 - i\vec{\sigma}\cdot\vec{\nabla}\phi_1 = E\phi_2\end{array}} \tag{12.31}$$

$\mathbf{H}$ commutes with $\vec{\mathbf{J}} = \vec{x}\times\vec{\mathbf{P}} + \frac{1}{2}\vec{\Sigma}$, i.e. with $\vec{\mathbf{J}}^2$ and J_3. Letting the space inversion operator Π act on Ψ gives

$$\Pi\Psi(\vec{x}) = \beta\Psi(-\vec{x})$$

So, $\Psi = \Psi_{j,m}$ is characterized by j and m, with $\vec{j} = \vec{L} + \frac{1}{2}\vec{\sigma}$ and formed out of Y and $\chi^{\pm}$. Since $j = l \pm \frac{1}{2}$, and the l-term gives a parity factor $(-1)^l$, we introduce $\Phi^{(\pm)}_{j,m}$. Equations (12.31) can be solved:

$$\boxed{E_{n,j} = \frac{m}{(1+e^4(n-(j+\frac{1}{2})+((j+\frac{1}{2})^2-e^4)^{1/2})^{-2})^{1/2}}} \tag{12.32}$$

with $n \leq j+\frac{1}{2}$ and $j = \frac{1}{2}, \frac{3}{2}, \ldots$. Thus the degeneracy in l is partially removed:

[figure]

Remarks

$\vec{J}$ commutes with $\mathbf{H}$, but in case $\vec{L}$ and $\vec{S}$ are coupled, $\vec{L}$ and $\vec{\Sigma}$ do not separately commute with $\mathbf{H}$. Therefore, it is preferable to consider the these commutators in the Fouldy-Wouthousen representation,

$$\begin{array}{rl}\vec{L}'_{\mathrm{D}} = & e^{-iS}\vec{L}_{\mathrm{FW}}e^{iS} \\ \vec{\Sigma}'_{\mathrm{D}} = & e^{-iS}\vec{\Sigma}_{\mathrm{FW}}e^{iS}\end{array} \tag{12.33}$$

where $\vec{L}_{\mathrm{FW}}$ and $\vec{\Sigma}_{\mathrm{FW}}$ each separately commute with $\mathbf{H}_{\mathrm{FW}} = \gamma^0 p_0$. Hence we can see that $\vec{L}'_{\mathrm{D}}$ and $\vec{\Sigma}'_{\mathrm{D}}$ also separately commute with $\mathbf{H}_{\mathrm{D}}$.

The relation to the total angular momentum is given by

$$\vec{L}'_{\mathrm{D}} + \frac{1}{2}\vec{\Sigma}'_{\mathrm{D}} = \vec{L}_{\mathrm{D}} + \frac{1}{2}\vec{L}_{\mathrm{D}} = \vec{J} \tag{12.34}$$

$$\vec{\Sigma}'_{\mathrm{D}} = \Sigma - \frac{i\gamma^0(\vec{\alpha}\times\vec{p})}{p_0} - \frac{\vec{p}\times(\vec{\Sigma}\times\vec{p})}{p_0(p_0+m)}$$

The helicity operator, $\vec{\Sigma}'_{\mathrm{D}}\cdot\vec{p} \stackrel{!}{=} \vec{\Sigma}\cdot\vec{p}$ commutes with $\mathbf{H}_{\mathrm{D}}$. Normalizing it by division by $|\vec{p}|$, we have

$$\left(\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}\right)^2 = 1$$

i.e., $\vec{\Sigma} \cdot \vec{p}/|\vec{p}|$ has eigenvalues ± 1 with eigenvectors $\chi_\pm$.

There also exists a split between the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ levels; this is the so-called *Lamb shift*, which one can try to explain by the zitterbewegung. We do not take this approach, but instead treat it in the context of 1-loop perturbation theory; see later.