# Quantum Field Theory II

by Prof. Michael G. Schmidt

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Supported by
Adisorn Adulpravitchai and Jenny Wagner

This is a first version of the lecture notes for professor Michael Schmidt's course on quantum field theory (part II), as taught in the 2007 summer semester. Comments and error reports are welcome; please send them to:

o.tieleman at students.uu.nl

# Contents

# Chapter 13

# Path integral formulation for fermions

In developing the path integral formulation for fermions, we should expect some difficulties, since in the canonical quantization procedure, we have seen anticommutators pop up where we had commutators in the bosonic case. In the path integral formulation, there are no operators, but only $c$ numbers, which commute. Since we want an anticommuting $\Psi_\alpha$, we need to develop a new mathematical tool: the anticommuting equivalent of scalars, called *Grassmann variables.*

## 13.1 Grassmann variables

Grassmann variables are usually denoted by Greek letters, $\theta$ being the most common one. They anticommute, so

$$\theta^2 = 0 \tag{13.1}$$

Also, since $\frac{d}{d\theta}$ is Grassmann valued,

$$\left(\frac{d}{d\theta}\right)^2 = 0 \tag{13.2}$$

Now consider

$$P(\theta) = \alpha + \beta\theta \tag{13.3}$$

where $\alpha$ is a normal scalar and $\theta$ and $\beta$ are Grassmann valued and hence anticommute. Note that this is the most general polynomial in $\theta$ possible, because $\theta^2 = 0$. Let the anticommutator

$$\left\{\frac{d}{d\theta}, \theta\right\} = \frac{d}{d\theta}\theta + \theta\frac{d}{d\theta} \tag{13.4}$$

act on $P(\theta)$:

$$\left(\frac{d}{d\theta}\theta + \theta\frac{d}{d\theta}\right)(\alpha + \beta\theta) = \frac{d}{d\theta}(\theta\alpha + \theta\beta\theta) + \theta\left(\frac{d}{d\theta}\alpha + \frac{d}{d\theta}(\beta\theta)\right) =$$
$$\alpha - \theta\beta = \alpha + \beta\theta$$

Here, we have used that $\frac{d}{d\theta}\theta = 1$, that $\frac{d}{d\theta}$ commutes with the non-Grassmann valued $\alpha$, and that it anticommutes with the Grassmann valued $\beta$ (i.e. $\frac{d}{d\theta}\beta = 0$). From this calculation, we can conclude that this anticommutator operates as the identity:

$$\left\{\frac{d}{d\theta}, \theta\right\} = 1$$

Integration over Grassmann variables must satisfy the following conditions:

$$\int d\theta = 0 \tag{13.5}$$

$$\int d\theta\theta = 1 \tag{13.6}$$

From this, we can see that for Grassmann variables, the operations differentiation and integration are equivalent. So, for example

$$\int d\theta P(\theta) = \alpha\int d\theta - \int d\theta\theta\beta = -\beta = \frac{dP(\theta)}{d\theta}$$

Otherwise, Grassmann integrals can be manipulated just like ordinary ones.

Now, consider a set of Grassmann variables $\theta_i$. Its members anticommute:

$$\{\theta_i, \theta_j\} = 0 \tag{13.7}$$

Since derivatives with respect to Grassmann variables are also Grassmann valued, we also have

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta_{ij} \qquad \text{and} \qquad \left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0 \tag{13.8}$$

Note that due to their anticommuting nature, the ordering is important, also under an integral:

$$\int d\theta_1\, d\theta_2\, \theta_1\theta_2 = -1 \qquad \text{whereas} \qquad \int d\theta_1\, d\theta_2\, \theta_2\theta_1 = 1 \tag{13.9}$$

### 13.1.1 Gaussian-like integral

For the path integral formulation of fermionic theory, we will need something like the Gaussian integral; it came up in the bosonic case as well. Here, we will work with integrals of the following type:

$$I_{2N}(M) = \int d\theta_1 \dots d\theta_{2N} \exp(-\theta^T M \theta) \tag{13.10}$$

where $M$ is an antisymmetric $2N \times 2N$-matrix and $\theta$ is a $2N$-dimensional Grassmann valued column vector.

Let us consider the case where $N = 2$. Since integration is the same as differentiation, and all $\theta$'s square to zero, the only possibility for a term to be nonzero is to have each $\theta_i$ exactly once under the integral. In this case, the only term fulfilling that requirement is the second-order term from the expansion of the exponential:

$$\frac{1}{2!}(\theta^T M \theta)^2 = \frac{2! \cdot 2^2}{2!} \theta_1\theta_2\theta_3\theta_4 (m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23})$$

The factor 2! in the numerator is there due to the fact that in each of the terms between the brackets, we can arrange the $m$'s in 2! ways; the $2^2$ is there because we can swap the indices on each of the $m$'s. Together, the terms between the brackets are just $(\det M)^{1/2}$, and integrating rids us of the $\theta$'s, so we are left with

$$I_4(M) = 2^2 (\det M)^{1/2}$$

Generalizing this to a case with unspecified $N$, we have to take the $N^{\text{th}}$-order term, which gives

$$I_{2N} = 2^N (\det M)^{1/2} \tag{13.11}$$

Now consider such integrals, modified by the presence of a 'source term' $\eta$, which is also a Grassmann valued column vector:

$$I_{2N}(M, \eta) = \int d\theta_1 \dots d\theta_{2N} \exp(-\theta^T M \theta + \eta^T \theta) \tag{13.12}$$

where, of course, $\{\eta_i, \theta_j\} = 0$. Completing the square in the exponent gives

$$I_{2N}(M, \eta) = \int d\theta'_1 \dots d\theta'_{2N} \exp\left(-\theta^{T\prime} M \theta' + \frac{1}{4}\eta^T M^{-1} \eta\right) = \\ \exp\left(\frac{1}{4}\eta^T M^{-1}\eta\right) 2^N (\det M)^{1/2}$$

### 13.1.2 Complex Grassmann variables

We will need complex Grassmann variables to represent fermions. Consider

$$\xi = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \qquad \text{and} \qquad \bar{\xi} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \tag{13.13}$$

Note that

$$d\theta_1 d\theta_2 = d\bar{\xi} d\xi \tag{13.14}$$

In preparation of the Gaussian-like integral for complex Grassmann variables, let us investigate the product $\theta^T M \theta$ with antisymmetric $M$. From the two-dimensional case, with $\theta = (\theta_1 \ \theta_2)^T$, we learn the following:

$$(\theta_1 \ \theta_2) \begin{pmatrix} m_{11} & m_{12} \\ -m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = -2i\bar{\xi} m_{12} \xi \tag{13.15}$$

This defines a new, anti-Hermitean, $N \times N$-matrix $\tilde{M}$, which in this case has dimension 1 and is given by

$$\tilde{M} = -2im_{ij} \tag{13.16}$$

Generalizing to the case of dimension $N$ gives

$$\begin{aligned} &\prod \int d\bar{\xi}_i \, d\xi_j \exp(-\bar{\xi}^T \tilde{M} \xi) = \\ &\qquad \prod_i \prod_j \int d\bar{\xi}_i \int d\xi_j \prod_{k=1}^{N} \left(1 - \sum_{l=1}^{N} \bar{\xi}_k \tilde{M}_{kl} \xi_l\right) = \\ &\qquad \sum_{\{j_1 \ldots j_N\}} \tilde{M}_{1j_1} \ldots \tilde{M}_{Nj_N} \prod_{i,j} \int d\bar{\xi}_i \int d\xi_j \ \bar{\xi}_1 \xi_{j_1} \ldots \bar{\xi}_N \xi_{j_N} \end{aligned}$$

where the sum is over all permutations of the $j_i$. The 1 in the integrand has dropped out, since Grassmann integration over non-Grassmann valued scalars gives zero. The product of integrals gives

$$\text{sign} \begin{pmatrix} 1 & \ldots & N \\ \vdots & & \vdots \\ j_1 & \ldots & j_N \end{pmatrix}$$

so the result is

$$\boxed{\prod \int d\bar{\xi}_i d\xi_j \exp(-\bar{\xi}^T \tilde{M} \xi) = \det \tilde{M}} \tag{13.17}$$

The complex, Grassmann valued, Gaussian-like integral with sources is given by

$$I_N(\tilde{M},\eta,\bar{\eta}) = \prod_{i=1}^{N}\prod_{j=1}^{N}\int d\bar{\xi}_i \int d\bar{\xi}_j \exp(-\bar{\xi}^T\tilde{M}\xi + \bar{\eta}\xi + \bar{\xi}\eta) \tag{13.18}$$

Completing the square again and using eq. (13.17), we obtain

$$\boxed{I_N(\tilde{M},\eta,\bar{\eta}) = \det\tilde{M}\exp(\bar{\eta}\tilde{M}^{-1}\eta)} \tag{13.19}$$

**Note**

Since these are no real Gaussian integrals, we can make the substitutions

$$\tilde{M} \to \frac{\hat{M}}{i} \qquad \text{and} \qquad \eta \to i\hat{\eta}$$

and get

$$I_N(\hat{M},\hat{\eta},\bar{\hat{\eta}}) = \det\frac{\hat{M}}{i}\exp\left(-\bar{\hat{\eta}}\left(\frac{\hat{M}}{i}\right)^{-1}\hat{\eta}\right) \tag{13.20}$$

## 13.2 Dirac field

Now, we come to the physics: we want to describe a Dirac field $\Psi_\alpha(x)$. This is a four-vector field of complex Grassmann valued $\theta_\alpha$, defined for all $x$ (in the discrete case, the $x$ becomes an index: $x \to x_i$, $\Psi_\alpha(x) \to \Psi_{i,\alpha}$). Since these fields are Grassmann valued, they anticommute:

$$\{\Psi_\alpha(x), \Psi_\beta(y)\} = 0 \tag{13.21}$$

Let us go directly to the generating functional formulation of the path integral for fermions. The free theory has

$$\begin{aligned} Z_0(\eta_\alpha,\bar{\eta}_\beta) = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, \exp\Big\{ i\int d^4x\, \big[\bar{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x) + \\ \bar{\eta}(x)\Psi(x) + \bar{\Psi}(x)\eta(x)\big] \Big\} \end{aligned} \tag{13.22}$$

$\hat{M}$ from eq. (13.20) can, by comparison, be seen to be

$$\hat{M} = (i\gamma^\mu\partial_\mu - m) \tag{13.23}$$

To find $\hat{M}^{-1}$, we use the fact that

$$(i\gamma^\mu\partial_\mu - m)(i\gamma^\mu\partial_\mu + m) = -\partial^2 - m^2$$

By definition, we know

$$\begin{aligned}\hat{M} S_{\mathrm{F}}(x-x') &= i\delta^4(x-x') \quad \text{and} \\ (\partial^2+m^2) D_{\mathrm{F}}(x-x') &= -i\delta^4(x-x')\end{aligned}$$

So,

$$\underbrace{(i\gamma^\mu\partial_\mu - m)}_{\hat{M}}\underbrace{(i\gamma^\mu\partial_\mu + m)D_{\mathrm{F}}}_{S_{\mathrm{F}}} = i\delta^4(x-x')$$

and hence

$$i\hat{M}^{-1} = S_{\mathrm{F}}(x-x') \tag{13.24}$$

This allows us to rewrite $Z_0(\eta_\alpha, \bar{\eta}_\beta)$ as

$$Z_0(\eta,\bar{\eta}) = \exp\left[-\int dx\, dx'\, \bar{\eta}(x) S_{\mathrm{F}}(x-x')\eta(x)\right] Z_0(0,0) \tag{13.25}$$

where $Z_0(0,0) = \det S_{\mathrm{F}}^{-1}$. Now, check that the two-point function $\langle 0|\, \mathrm{T}(\Psi_\alpha(y)\bar{\Psi}_\beta(x))\, |0\rangle$ indeed gives back $S_{\mathrm{F}}$:

$$\begin{aligned}\langle 0|\, \mathrm{T}(\Psi_\alpha(y)\bar{\Psi}_\beta(x))\, |0\rangle &= \frac{1}{Z_0(0,0)}\left(\frac{1}{-i}\frac{\delta}{\delta\bar{\eta}_\alpha(y)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\eta_\beta(x)}\right) Z_0(\eta,\bar{\eta})\Big|_{\eta=\bar{\eta}=0} \\ &= S_{\mathrm{F}}(x-y) \end{aligned}\tag{13.26}$$

### 13.2.1 Yukawa coupling

Remember the interaction Lagrangians for Yukawa coupling:

$$\begin{aligned}\mathcal{L}_{\text{int}} &= -f\bar{\Psi}(x)\Psi(x)\Phi(x) \qquad \text{for a real scalar field } \Phi \\ \mathcal{L}_{\text{int}} &= -if\bar{\Psi}(x)\gamma_5\Psi(x)\Phi(x) \qquad \text{for a pseudoscalar field } \Phi\end{aligned}\tag{13.27}$$

The generating functional now has functional derivatives with respect to $\eta$ and $\bar{\eta}$, for the $\Psi$- and $\bar{\Psi}$-fields, but also with respect to $j(x)$, for the $\Phi$-field:

$$Z[\eta,\bar{\eta},j] = \exp\left[i\int d^4y \mathcal{L}_{\text{int}}\left(\frac{1}{-i}\frac{\delta}{\delta\eta}, \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}, \frac{1}{i}\frac{\delta}{\delta j}\right)\right] Z_0(\eta,\bar{\eta})Z_0(j) \tag{13.28}$$

Expanding the exponential produces the Feynman rules for this theory (see exercise sheet 2). Note that the $\mathcal{L}_{\text{int}}$'s of eqs. (13.27) are real if the complex conjugation operation on a product of complex Grassmann variables is defined to exchange order like Hermitean conjugation (which here is only with respect to Dirac indices).

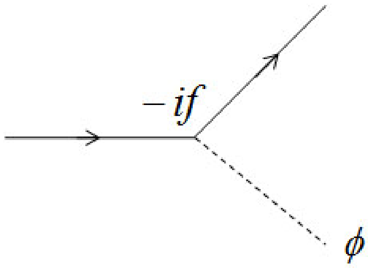



Figure 13.1: Yukawa interaction

**Note**

Closed fermion lines give an additional factor of $-1$: in the functional derivatives

$$\frac{\delta}{\delta\bar{\eta}_i(z)}\frac{\delta}{\delta\eta_j(z)}\frac{\delta}{\delta\bar{\eta}_k(z')}\frac{\delta}{\delta\eta_l(z')}$$

one always has to contract $\eta$'s with $\bar{\eta}$'s, which in this case requires a commutation to ensure the right ordering. Since the $\eta$'s are Grassmann valued, they anticommute, giving rise to a minus sign. The integrand then becomes

$$-S_{il}(z-z')S_{kj}(z'-z)$$

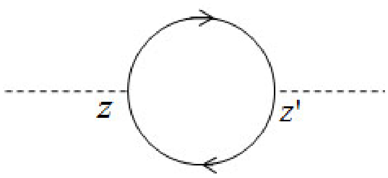



Figure 13.2: Closed fermion line

### 13.2.2 QED

Coupling fermion fields to the electromagnetic vector potential is done by so-called *minimal coupling*, or introducing the *covariant derivative* $\mathcal{D}$:

$$\partial_\mu \to \partial_\mu + ieA_\mu$$

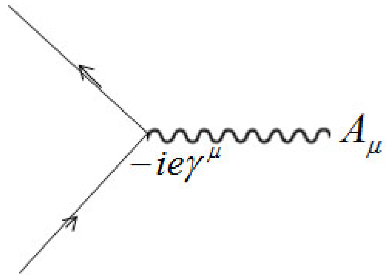



Figure 13.3: Minimal coupling

The electromagnetic Lagrangian then has the form

$$\mathcal{L}_{\rm em} = \bar{\Psi}(i\gamma^\mu \underbrace{(\partial_\mu + ieA_\mu)}_{\mathcal{D}_\mu})\Psi \tag{13.29}$$

The interaction Lagrangian is given by

$$\mathcal{L}_{\rm int} = -e\bar{\Psi}\gamma^\mu\Psi A_\mu \tag{13.30}$$

# Chapter 14

# Feynman loop diagram integrals, regularization and renormalization

## 14.1 Recapitulation of the Feynman rules

A short reminder: Feynman rules for $\Phi^4$-theory.

First, draw all possible diagrams connecting the outer points of the T-product $\langle 0|\, \mathrm{T}(\Phi(x_1)\ldots\Phi(x_n))\,|0\rangle$ to be calculated, with a certain number of vertices (i.e., to a certain order in $\lambda$, the coupling constant). Then write down the corresponding expression; in position space, this consists of:

- a factor $iG_{\mathrm{F}}(x_i - x_k)$ for each propagator (line)
- a factor $-i\lambda$ for each vertex; $\lambda$ is the coupling strength
- statistical factors correcting for overcounting
- integration over vertices $y$: $\int d^4y$

After a Fourier transform, i.e. going to momentum space, these rules become:

- a factor $i/(k^2 - m^2 + i\epsilon)$ for each propagator
- a factor $-i\lambda$ for each vertex
- statistical factors correcting for overcounting
- delta functions ensuring 4-momentum conservation at vertices
- integration over independent loop momenta: $\int \frac{d^4k}{(2\pi)^4}$
- an overall momentum conservation factor $(2\pi)^4\delta^4(\sum p)$

In both cases, one can apply the LSZ reduction formalism: this truncates outer lines and produces factors of $(\sqrt{Z})^{-1}$.

## 14.2 Calculation of a Feynman graph with loops

### 14.2.1 Recipe

So far, in our discussion of the Feynman rules, we have avoided calculating any graphs with loops in them. Let us remedy this deficit here. We will work by the following recipe:

(i) Derive the expression corresponding to the graph

(ii) Apply Feynman's trick to deal with the various denominators from the propagators (we will work in momentum space)

(iii) Shift the inner (loop) momenta to get rid of the terms that are linear in these momenta in the Feynman denominator

(iv) Wick rotate, both inner and outer variables, to avoid the poles in the integral

(v) Rewrite the momentum integral into an angular integral and one over the magnitude of the momentum; perform the angular integration

(vi) Introduce a cut-off $L^2$, also called *regulator*

(vii) Calculate the integral for large $L^2$

(viii) Reverse the Wick rotation, i.e., go back to Minkowski space

(ix) Subtract and add the diagram at some loop momentum configuration to isolate the $L^2$-dependent divergent part

### 14.2.2 Calculation

As usual, we will take $\Phi^4$-theory as example; we will calculate the graph in figure 14.1.

Step (i): from the Feynman rules, we get the following expression:

$$\frac{1}{\sqrt{Z}^4}\frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k^2-m^2+i\epsilon)}\frac{i}{((k-q)^2-m^2+i\epsilon)} \tag{14.1}$$

where, due to conservation of momentum,

$$p_1+p_2=q=p_1'+p_2'$$

and we have chosen one of the inner momenta to be $k$ (forcing the other one to be $q-k$).

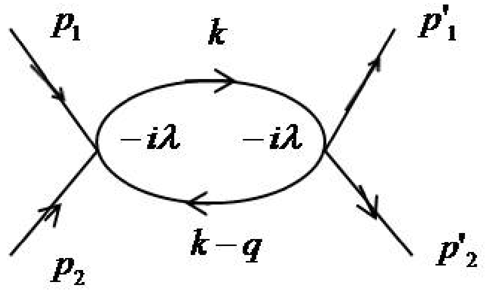



Figure 14.1: Graph to be calculated

Step (ii): to calculate expression (14.1), we will need the following mathematical identity:

$$\frac{1}{AB} = \int_0^1 d\alpha \frac{1}{(\alpha A + (1-\alpha)B)^2} \tag{14.2}$$

This allows us to rewrite expression (14.1) as

$$\frac{1}{Z^2}\frac{\lambda^2}{2(2\pi)^4}\int d^4k \int_0^1 d\alpha[\alpha k^2 + (1-\alpha)(k-q)^2 - m^2 + i\epsilon]^{-2}$$

Step (iii): we can rewrite the part between brackets by defining $k'$:

$$\begin{aligned} k' &:= k - (1-\alpha)q, \qquad \text{so} \\ \alpha k^2 + (1-\alpha)(k-q)^2 &= k^2 - 2(1-\alpha)q\cdot k + (1-\alpha)q^2 = \\ &k'^2 - (1-\alpha)^2 q^2 + (1-\alpha)q^2 = \\ &k'^2 - \alpha(1-\alpha)q^2 \end{aligned}$$

Step (iv): now, to avoid the poles, where the integrand is divergent (hence the $i\epsilon$-prescription), we will transform to Euclidean coordinates. This involves the following transformations:

$$\begin{aligned} x^j_{\mathrm{E}} &= x^j; \qquad x_4 = ix_0 \\ k^j_{\mathrm{E}} &= -k^j; \qquad k_4 = -ik_0 \qquad \text{and thus} \\ \exp(i(k_0x_0 - \vec{k}\cdot\vec{x})) &= \exp(i(k_4x_4 - \vec{k}\cdot\vec{x})) = \exp(ik_{\mathrm{E}}\cdot x_{\mathrm{E}}) \end{aligned}$$

In these coordinates, the integral becomes

$$iC\int_0^1 d\alpha \int d^4k'_{\mathrm{E}}[-k_4'^2 - \vec{k}'^2 + \alpha(1-\alpha)q^2 - m^2]^{-2}$$

this time without $i\epsilon$, as we do not hit the poles, and hence do not need to decide how to go around them. The constants in front of the integral have been gathered into $C$.

Step (v): now notice that the integrand does not depend on the direction of $k'_{\rm E}$, and hence, we can convert the integral over $d^4k'_{\rm E}$ into one over $\frac{1}{2}{k'_{\rm E}}^2 d{k'_{\rm E}}^2$ and one over the 4-dimensional equivalent of solid angle:

$$\int d^4k'_{\rm E} \to \int d\Omega_4 \frac{{k'_{\rm E}}^2 d{k'_{\rm E}}^2}{2} = \frac{2\pi^2}{2}\int_0^\infty {k'_{\rm E}}^2 d{k'_{\rm E}}^2$$

Completing the transfer to Euclidean coordinates by replacing $q^2$ with $-q_{\rm E}^2$, and absorbing the prefactors (including the minus sign) from the angular integration into $C'$, we are left with

$$C'\int_0^1 d\alpha \int_0^\infty d{k'_{\rm E}}^2 \left\{ {k'_{\rm E}}^2[{k'_{\rm E}}^2 + \alpha(1-\alpha)q_{\rm E}^2 + m^2]^{-2} \right\}$$

which is logarithmically divergent.

Step (vi): to see this more clearly, let us introduce a "cut-off", also called *regulator*, $L^2$, as upper integration limit for the ${k'_{\rm E}}^2$-integral. Then, by partial integration, we get

$$C'\int_0^{L^2} d{k'_{\rm E}}^2 \left\{ {k'_{\rm E}}^2[{k'_{\rm E}}^2 + \alpha(1-\alpha)q_{\rm E}^2 + m^2]^{-2} \right\} =$$

$$C'\frac{-{k'_{\rm E}}^2}{{k'_{\rm E}}^2 + \alpha(1-\alpha)q_{\rm E}^2 + m^2}\Bigg|_0^{L^2} + C'\underbrace{\int_0^{L^2}[{k'_{\rm E}}^2 + \alpha(1-\alpha)q_{\rm E}^2 + m^2]^{-1}}_{\log({k'_{\rm E}}^2 + \alpha(1-\alpha)q_{\rm E}^2 + m^2)\,\big|_0^{L^2}}$$

and our expression takes the following form

$$C'\int_0^1 d\alpha \left\{ \log\left(\frac{L^2 + \alpha(1-\alpha)q_{\rm E}^2 + m^2}{\alpha(1-\alpha)q_{\rm E}^2 + m^2}\right) - \frac{L^2}{L^2 + \alpha(1-\alpha)q_{\rm E}^2 + m^2} \right\}$$

Step (vii): for large $L^2$, the logarithmic term becomes $\log(L^2/(\alpha(1-\alpha)q_{\rm E}^2 + m^2))$, and the second term goes to $-1$. Since this is the limit we want to take (remember, $L^2$ came in the place of $\infty$), we can conclude that this expression diverges logarithmically.

Step (viii): now, we substitute back $q_{\rm E}^2 \to -q^2$, and subtract and add this expression evaluated at some momentum configuration to obtain a finite expression plus some divergent term. For practical reasons, we will choose this configuration to be the symmetrical point, where $q^2 = u = t = s = 4m^2/3$. With this, we get

$$\Lambda^{(2)}_{Sym} + \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha \left\{ \log\frac{L^2}{-\alpha(1-\alpha)q^2 + m^2} - \log\frac{L^2}{-\alpha(1-\alpha)(4m^2/3) + m^2} \right\} \tag{14.3}$$

$$= \Lambda^{(2)}_{Sym} + \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha \left\{ \log\frac{\alpha(1-\alpha)(4m^2/3) + m^2}{\alpha(1-\alpha)q^2 + m^2} \right\} \tag{14.4}$$

The $\log L^2$ terms cancel, so the integral is finite for $L^2 \to \infty$. The $L^2$-dependence has effectively been hidden in the $\Lambda^{(2)}$-term, which is the result of correcting for the subtraction. We will return to this point in greater detail in the context of the renormalization program.

## 14.3 Calculation of general Feynman graph expressions

Generalizing the above to a Feynman graph with $k$ vertices and some number of loop momenta $l_i$ and momentarily forgetting about prefactors, we want to evaluate the following expression:

$$I = \int \frac{d^4 l_1}{(2\pi)^4} \cdots \frac{d^4 l_k}{(2\pi)^4} \prod_{j=1}^{n} \frac{1}{p_j - m_j + i\epsilon} \tag{14.5}$$

with momentum conservation at the vertices as additional constraint. Feynman's trick for $n$ factors $A_i^{-1}$ looks as follows:

$$(A_1 \ldots A_n)^{-1} = (n-1)! \int_0^1 d\alpha_1 \ldots d\alpha_n \frac{\delta(1 - \alpha_1 - \alpha_2 - \cdots - \alpha_n)}{(A_1\alpha_1 + A_2\alpha_2 + \cdots + A_n\alpha_n)^n} \tag{14.6}$$

To prove this, one can simply perform the integral, or use the following:

$$\prod_{j=1}^{n} \frac{1}{a_j + i\epsilon} = (i)^{-n} \int_0^\infty d\alpha_1 \ldots d\alpha_n \exp\left[i \sum \alpha_j a_j - \epsilon \left(\sum \alpha_j\right)\right] \times \underbrace{\int_0^\infty \frac{d\lambda}{\lambda} \delta\left(1 - \sum \frac{\alpha_j}{\lambda}\right)}_{1}$$

Now substitute $\alpha_j \to \lambda\alpha_j$:

$$\int_0^\infty \frac{d\lambda}{\lambda} \lambda^n \exp\left\{\lambda \left[i \sum \alpha_j a_j - \epsilon \left(\sum \alpha_j\right)\right]\right\}$$

and use $\int_0^\infty d\lambda \lambda^n = \Gamma(n+1)$ to get the result above (also see problem sheet 4). Note: one may also write the left-hand side of eq. (14.6) as an integral over a "polyhedron":

$$\int_0^1 d\alpha_1 \ldots d\alpha_n \delta(1 - \alpha_1 - \alpha_2 - \cdots - \alpha_n) = \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \ldots \int_0^{1-\alpha_1-\cdots-\alpha_{n-2}} d\alpha_{n-1}$$

Continuing with our integral, we now have

$$I = (n-1)! \int \frac{d^4 l_1}{(2\pi)^4} \cdots \frac{d^4 l_k}{(2\pi)^4} \int_0^1 d\alpha_1 \dots d\alpha_n \frac{\delta(1-\alpha_1-\cdots-\alpha_n)}{\left(\sum \alpha_j (p_j^2 - m_j^2 + i\epsilon)\right)^n}$$

Let us investigate the denominator a bit more closely. First, we can rewrite the $p_j$ as

$$p_j = k_j + \sum_{r=1}^{n} \eta_{jr} l_r$$

where $k_j$ is a combination of the outer momenta, and $\eta_{jr}$ a prefactor:

$$\eta_{jr} = \begin{cases} 1 & \text{if } p_j \text{ is in the } r\text{-loop and } p_j \parallel l_r \text{ with same sign} \\ -1 & \text{if } p_j \text{ is in the } r\text{-loop and } p_j \parallel l_r \text{ with different sign} \\ 0 & \text{otherwise} \end{cases}$$

Plugging this in, the denominator becomes

$$\underbrace{\sum_{j,r,r'} \alpha_j \eta_{jr} \eta_{jr'}}_{Z_{rr'}} l_r l_{r'} + 2 \sum_{j,r} k_j \alpha_j \eta_{jr} l_r + \sum_j (k_j^2 - m_j^2) \alpha_j$$

Now, we shift our momenta: $l_r \to l'_r$, such that the linear $l$-term vanishes. $k'_r$ then depends on the outer momenta *and* on the $\alpha_i$. The new $l'_r$ must fulfill the following conditions:

(i) $\sum_{j=1}^{n} k'_j \alpha_j \eta_{jr} = 0$ for each loop $r$

(ii) The relation to the outer momenta is determined via momentum conservation at the vertices.

The matrix $Z_{rr'}$ can be diagonalized; then, the $l'_r$ can be integrated over, using

$$d^4 l' = l'^2 dl'^2 \frac{d\Omega_4}{2}, \qquad \int d\Omega_4 = 2\pi^2 \quad \text{and}$$

$$\frac{2\pi^2}{2} \frac{1}{(2\pi)^4} \int_0^\infty \frac{dl'^2 l^2}{(Al^2+B)^n} = \frac{1}{A^2} \frac{1}{B^{n-2}} \frac{1}{(n-1)(n-2)} \frac{1}{16\pi^2}$$

This brings us to the final form of our integral:

$$I = \int_0^1 d\alpha_1 \dots d\alpha_n \delta\left(1 - \sum \alpha_j\right) (\det Z)^{-2} \left[\sum (k_j'^2 - m_j^2)\alpha_j\right]^{-n+2k} \times \left(\frac{1}{16\pi^2}\right)^k \frac{(n-1)!}{(n-1)\dots(n-2k)} \tag{14.7}$$

**Remark**

Conditions (i) and (ii) are typical of circuit systems (see Björken and Drell, or C.S.Lam, *Phys. Rev.* **D48** (1993) 873). $k'_j$ then corresponds to the current in line $j$, and $\alpha_j$ to the resistance. Kirchhoff's laws also hold:

(i) $\sum_{j=1}^{n} k'_j \alpha_j \eta_{jr} = 0$ for each loop $r$ corresponds to $\sum U = 0$ in each loop

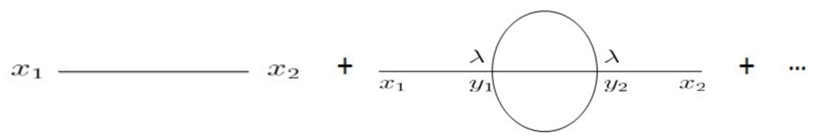
(ii) The sum of incoming momenta at vertex $(i)$, $\sum k_j^{(i)} + q^{(i)} = 0$ corresponds to charge conservation at vertices: $\sum I = 0$. (Note that the $l$ cancel in the first sum).

We even have Ohm's law: $IR = V$, where $I \triangleq k$, $R \triangleq \alpha$ and $V \triangleq \Delta x$:

$$\Delta x^\mu = k^\mu \alpha$$

## 14.4 Regularization and renormalization in a simple case

Let us consider a 2-point function, $\langle 0| \mathrm{T}(\Phi(x_1)\Phi(x)2) |0\rangle$ in $\Phi^4$-theory, to order $\lambda^2$. In Feynman diagrams, this means:




(Here, we have removed the tadpole graphs by requiring normal ordering in $\mathbf{H}_{\mathrm{int}}$; we will come back to this later.) The corresponding expression is:

$$iD_{\mathrm{F}}(x_1 - x_2) + \int d^4y_1 d^4y_2 iD_{\mathrm{F}}(x_1 - y_1)\frac{(-i\lambda)^2}{3!}(iD_{\mathrm{F}}(y_1 - y_2))^3 iD_{\mathrm{F}}(y_2 - x_2) + \ldots \tag{14.8}$$

The combinatorial factor 1/3! is to correct for the fact that there are 3! permutations of the loop propagators, which are all counted separately. Applying a Fourier transform, i.e. going to momentum space, we get out usual momentum conservation factor of $(2\pi)^4\delta^4(p_1 + p_2)$. We will leave out this factor here, and write

$$p_1 = -p_2 = p$$

We will write the Feynman propagator

$$\tau(p, -p) = \frac{1}{p^2 - m_0^2 + i\epsilon}$$

with $m_0$, for later convenience; $m_0$ is called "bare mass". Expression (14.8) then becomes

$$\frac{i}{p^2 - m_0^2 + i\epsilon} + \frac{(-i\lambda)^2}{3!}\left(\frac{i}{p^2 - m_0^2 + i\epsilon}\right)^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \times$$
$$\frac{i}{(k_1^2 - m_0^2 + i\epsilon)} \frac{i}{(k_2^2 - m_0^2 + i\epsilon)} \frac{i}{((k_1 + k_2 - p)^2 - m_0^2 + i\epsilon)} + \ldots$$

To make our goal more explicit, we now rewrite the second term as follows:

$$\frac{i}{(p^2 - m_0^2 + i\epsilon)} \left(-i\Sigma^{(2)}(m_0^2, p^2)\right) \frac{i}{(p^2 - m_0^2 + i\epsilon)}$$

The middle factor represents the "self-energy" of the particle. By simply counting powers of $k$, one can argue that this must be quadratically divergent. However, a divergent expression does not make sense physically, so we will have to *regularize* it. In our simple example before, we used the so-called *cut-off regularization*; we will discuss general regularization in detail later on. Of course, having a cut-off requires physical explanation: the physics at very large momenta (corresponding to very small distances) is not known, and by introducing a cut-off, we try to simply throw this away. Luckily, we will find that after some redefinitions, a procedure called *renormalization*, we will be able to remove the cut-off without changing physical quantities. The procedure of regularization and renormalization could be seen as an attempt to hide our ignorance of the physics at very short distances in a few parameters, namely the *renormalization pieces* we will encounter later on.

**Exercise**: calculate $\Sigma^{(2)}$ (see dimensional regularization, further on).

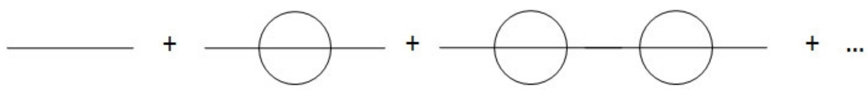
Taylor expanding the regularized $\Sigma^{(2)}(m_0^2, p^2, L)$ at $p^2 = m^2$ gives:

$$\Sigma^{(2)}(m_0^2, p^2, L) = \Sigma^{(2)}(m_0^2, m^2, L) + (p^2 - m^2)\, \frac{d}{dp^2}\Sigma^{(2)}(m_0^2, m^2, L)\bigg|_{p^2=m^2} + \Sigma^{(2)}_{\text{rest}} \tag{14.9}$$

(N.b.: $m^2$ is not necessarily $m_0^2$; in case it is, one speaks of "bare perturbation theory".) The first of these terms is quadratically divergent as $L \to \infty$, the second is logarithmically divergent, and the rest is convergent.

### Interpretation of self-energy

If we let go of the 1PI-requirement and insert more loops,



we obtain a geometrical series (assuming the self-energy is small):

$$\left(\frac{i}{p^2 - m_0^2 + i\epsilon}\right)\Bigg[\ 1 + (-i)\Sigma^{(2)}(m_0^2, p^2, L)\frac{i}{p^2 - m_0^2 + i\epsilon} + \\ (-i)^2\left(\Sigma^{(2)}(m_0^2, p^2, L)\frac{i}{p^2 - m_0^2 + i\epsilon}\right)^2 + \dots\ \Bigg] = \\ \frac{i}{p^2 - m_0^2 + i\epsilon} \times \frac{1}{1 + i\Sigma^{(2)}(m_0^2, p^2, L)\frac{i}{p^2 - m_0^2 + i\epsilon}} = \frac{i}{p^2 - m_0^2 - \Sigma^{(2)}(m_0^2, p^2, L)}$$

This is equivalent to the old propagator with a pole shifted to some physical mass $m_{\mathrm{p}}^2$ and some change in normalization. In other words, the self-energy can be seen as shifting the mass and changing the normalization of the field (see below).

**Renormalization**

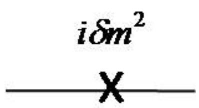
Let us reformulate our problem: instead of the bare mass $m_0^2$, we will use the above-mentioned physical mass $m_{\mathrm{p}}$, defined by the pole position; we will denote it as

$$\boxed{m_{\mathrm{p}}^2 = m_0^2 + \delta m^2} \tag{14.10}$$

We will have to reformulate our Lagrangian, by adding and subtracting terms of $\frac{1}{2}\delta m^2 \Phi^2$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m_{\mathrm{p}}^2 \Phi^2 + \frac{1}{2}\delta m^2 \Phi^2 \tag{14.11}$$

This last term, called "counterterm" for obvious reasons, can be considered a part of the interaction ($\mathcal{L}_{\mathrm{int}}$). This means that it has the role of a vertex in the Feynman rules:




It has a factor 2 for combinatorial reasons, which nicely cancels the factor of $\frac{1}{2}$ from the Lagrangian. $\delta m^2$ is of order $\mathcal{O}(\lambda^2)$ in a perturbative expansion.

It is very natural, although of course by no means necessary, to expand expression (14.9) around $m^2 = m_\mathrm{p}^2$. If we do this, we obtain the following propagator:

$$i\left[p^2 - m_\mathrm{p}^2 - \Big(\ \Sigma^{(2)}(m_\mathrm{p}^2, m_\mathrm{p}^2, L) - \delta m^2(L)\right.$$
$$\left.(p^2 - m_\mathrm{p}^2)\Sigma^{(2)\prime}(m_\mathrm{p}^2, m_\mathrm{p}^2, L) +\ \Sigma^{(2)}_\mathrm{rest}(m_\mathrm{p}^2, p^2, L)\Big)\right]^{-1}$$

where $\Sigma^{(2)}_\mathrm{rest}$ is of order $\mathcal{O}((p^2 - m_\mathrm{p}^2)^2)$. We postulated the pole at $p^2 = m_\mathrm{p}^2$, so we can see that

$$\delta m^2(L) = \Sigma^{(2)}(m_\mathrm{p}^2, m_\mathrm{p}^2, L)$$

which reduces the propagator to

$$i\left[p^2 - m_\mathrm{p}^2 - (p^2 - m_\mathrm{p}^2)\Sigma^{(2)\prime}(m_\mathrm{p}^2, m_\mathrm{p}^2, L) - \Sigma^{(2)}_\mathrm{rest}\right]^{-1} =$$
$$i\left[1 - \Sigma^{(2)\prime}(m_\mathrm{p}^2, m_\mathrm{p}^2, L) - \Sigma^{(2)}_\mathrm{rest}/(p^2 - m_\mathrm{p}^2)\right]^{-1}[p^2 - m^2]^{-1}$$

of which $-\Sigma^{(2)}_\mathrm{rest}/(p^2 - m_\mathrm{p}^2)$ vanishes for $p^2 = m_\mathrm{p}^2$.

**Wave function renormalization**

This whole discussion started with calculating a normal 2-point function $\langle 0|\,\mathrm{T}(\Phi(x_1)\Phi(x_2))\,|0\rangle$, where the $\Phi$'s are interacting fields. Recall the Yang-Feldman equation from chapter 5:

$$\sqrt{Z}\Phi_\mathrm{in}(x) = \Phi(x) - \int d^4y\, G_\mathrm{ret}(x-y, m)\, j(y)$$

where $\Phi_\mathrm{in}$ has the free field propagator. Renaming our interacting field $\Phi$ to $\Phi_0$, i.e. the unrenormalized, "bare", field, we write

$$\boxed{\Phi_\mathrm{ren} = \frac{1}{\sqrt{Z}}\Phi_0} \tag{14.12}$$

and see that $\Phi_\mathrm{ren}$ also has the free field propagator, but only around $p^2 = m^2$. After this rewriting, the 2-point function becomes

$$\langle 0|\,\mathrm{T}(\Phi_0(x_1)\Phi_0(x_2))\,|0\rangle = Z\,\langle 0|\,\mathrm{T}(\Phi_\mathrm{ren}(x_1)\Phi_\mathrm{ren}(x_2))\,|0\rangle$$

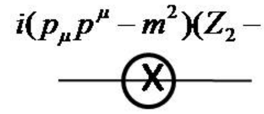
and we see

$$\boxed{Z = \left[1 - \Sigma^{(2)\prime}(m^2, m^2, L)\right]^{-1}} \tag{14.13}$$

This is called "wave function renormalization". $Z$ is, by convention, called $Z_2$. This calls for another reformulation of the Lagrangian:

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{2}\partial_\mu \Phi_0 \partial^\mu \Phi_0 - m_0^2 \Phi_0^2 - \frac{\lambda_0}{4!}\Phi_0^4 = \\
& \frac{1}{2}\partial_\mu \Phi_{\text{ren}} \partial^\mu \Phi_{\text{ren}} - m^2 \Phi_{\text{ren}}^2 - \frac{\lambda_0}{4!} Z_2^2 \Phi_{\text{ren}}^4 + \\
& \frac{1}{2}\delta m^2 Z_2 \Phi_{\text{ren}}^2 + \frac{1}{2}(\partial_\mu \Phi_{\text{ren}} \partial^\mu \Phi_{\text{ren}} - m^2 \Phi_{\text{ren}}^2)(Z_2 - 1)
\end{aligned}
\tag{14.14}
$$

The last term is a new counterterm, which again is treated as part of the interaction, and becomes a vertex with corresponding coupling constant in the diagram:




**Remarks**

- $\Sigma^{(2)\prime}$ is logarithmically divergent for $L \to \infty$, and in this limit, $Z \to 0$. $Z \approx 1 + \Sigma^{(2)\prime}$ (to order $\lambda^2$) would not allow this.
- In old literature, calculations may be based on bare quantities instead of renormalized ones. The counterterm procedure has the advantage that one deals with physical quantities only, and that e.g. the self-energy geometrical series is under control. In general, one should also pay attention to remarks about notation conventions.
- Our notation convention: we will denote bare quantities by a subscript "0" (e.g. $\Phi_0$), and renormalized ones *without* subscript (e.g. $\Phi$). So, the subscript "ren" will be dropped.
- By this renormalization, we get rid of the factors of $1/\sqrt{Z}$ from the LSZ-reduction formula.
- The Klein-Gordon operator $(\partial^2 + m^2)$ with physical mass $m$ cancels the pole in the outer propagator.
- **Exercise**: calculate the self-energy of Yukawa-particles.

### 14.4.1 4-point function

We still have to take into account the divergence of the 4-point function from the beginning of this chapter. To order $\mathcal{O}(\lambda^2)$, we had hidden it in

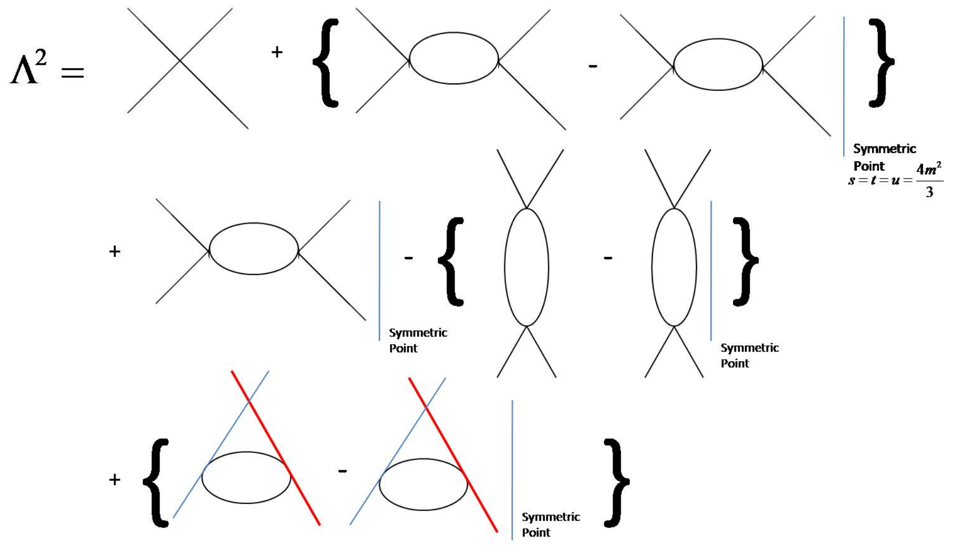



Figure 14.2: Divergence of 4-point function

$\Lambda^{(2)}$ (see figure 14.4.1). To deal with the divergence of the 4-point function, we have to address the renormalization of the third term of the right-hand side of eq. (14.14). Momentarily forgetting about the other counterterms, which are normally included in $\mathcal{L}_{\text{int}}$, we have

$$\mathcal{L}_{\text{int}} = -\frac{\lambda_0}{4!} Z_2^2 \Phi^4 = -\frac{\lambda}{4!}\Phi^4 + \frac{\delta\lambda}{4!}\Phi^4 \tag{14.15}$$

i.e., we get another counterterm. Like in the case of the 2-point function, we can tune this to cancel the contribution at some point of our choice, e.g. the symmetric point:

$$\Lambda^{(2)}_{\text{Sym}} = i\lambda$$

**Notation: multiplicative renormalization**

Multiplicative renormalization is a notation scheme in which the $\delta$-terms are written as multiples of their bare counterparts. One writes

(i) $\lambda = \lambda_0 Z_2^2 / Z_1$, which defines $Z_1$. Then,

$$\lambda_0 Z_2^2 \Phi^4 = \lambda_0 \frac{Z_2^2}{Z_1}\Phi^4 - \underbrace{\lambda_0 Z_2^2}_{\lambda Z_1}\left(\frac{1}{Z_1} - 1\right)\Phi^4 =$$
$$(\lambda - \lambda(1 - Z_1))\Phi^4 = (\lambda - \delta\lambda)\Phi^4$$

(ii) $\delta m^2 Z_2 - m^2(Z_2 - 1) = -m^2(Z_0 - 1)$ with $m^2 Z_0 = m_0^2 Z_2$, so $m^2 = (Z_2/Z_0)m_0^2$

(iii) The total set of counterterms can also be written as follows (see e.g. Peskin & Schröder):

$$\frac{1}{2}\underbrace{(Z_2 - 1)}_{\delta_Z}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\delta_m\Phi_{\text{ren}}^2 - \underbrace{(Z_1 - 2)}_{\delta_\lambda}\frac{\lambda_{\text{ren}}}{4!}\Phi_{\text{ren}}$$

Putting all of the above together and using Itzykson & Zuber's convention, our Lagrangian ends up in the following form:

$$\begin{aligned}\mathcal{L} =&\mathcal{L}_0 + \mathcal{L}_{\text{int}} = \frac{1}{2}(\partial_\mu\Phi_0\partial^\mu\Phi_0 - m_0^2\Phi_0^2) - \frac{\lambda_0}{4!}\Phi_0^2 = \\ &\frac{1}{2}(\partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2) - \frac{\lambda}{4!}\Phi^4 + \\ &\frac{\lambda}{4!}(1 - Z_1)\Phi^4 + \frac{1}{2}(Z_2 - 1)(\partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2) + \frac{1}{2}Z_2\delta m^2\Phi^2\end{aligned} \tag{14.16}$$

The last line contains all the counterterms, which we will, in our lowest order calculation (without tadpoles), approach perturbatively to order $\lambda$ (the first one) or $\lambda^2$ (the last two). Note that it is also possible to consider the $p^2$- and $m^2$-counterterms separately (option (iii) above).

We have three renormalization parameters, $Z_1$, $Z_2$ and $\delta m^2$. These are fixed by the *physical renormalization conditions*:

$$\tau(p, -p)^{-1}\Big|_{p^2=m^2} = 0 \tag{14.17}$$

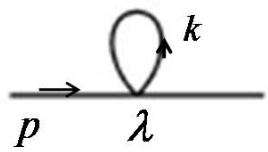
$$\frac{d}{dp^2}\left(\tau(p, -p)\right)^{-1}\Big|_{p^2=m^2} = \frac{1}{i} \tag{14.18}$$

$$\Lambda_{\text{Sym}} = i\lambda \tag{14.19}$$

where, logically, the first two apply to 2-point functions, and the last one to 4-point functions. Later on we will see that these conditions are the only ones needed for renormalizable theories (like $\Phi^4$-theory), also when taking into account higher loop orders. The counterterms are then arranged to be power series in $\lambda$, such that they cancel the divergences order by order.

### Note

In this chapter, we have neglected tadpole diagrams (like e.g. Itzykson and Zuber). This corresponds to requiring the interaction term in the Hamiltonian to be normal ordered, which is somewhat unnatural in the path integral approach. Peskin and Schröder, for example, do treat these diagrams carefully (see also the problem sheets). In brief, this is what happens:

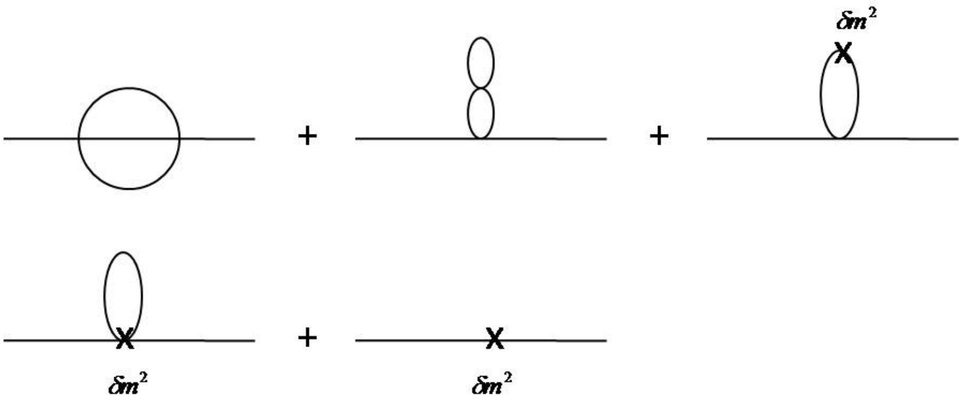



- One already has a self-energy at order $\lambda$, which is *independent of the outer momentum* $p$:

$$\sim -i\lambda \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}$$

It is quadratically divergent, and requires an addition to the mass-counterterm of order $\lambda$.

- To order $\lambda^2$, we have some additional 1-particle irreducible graphs (again, see problem sheets):




Still, however, it is only the mass-renormalization that changes; the wavefunction remains as it is.

# Chapter 15

# Regularizations and renormalization conditions

## 15.1 More general regularizations

### 15.1.1 Preliminary mathematics

Products of distributions are not well-defined. Yet, in perturbation theory, we do encounter them: propagators are distributions, for which we need a proper definition. To find this, we will consider distributions as limits of regular functions:

$$D(x) = \lim_{n\to\infty} D_n^{\mathrm{reg}}(x) \tag{15.1}$$

**Definition**: let $\rho_N(x_1,\dots,x_n)$ be in the linear subspace $S_N$ of the test function space $S$ of $\rho(x_1,\dots,x_n)$, with functions vanishing to order $N$ if any two arguments coincide. Then, the following lemma holds:

**Lemma**: let

$$\prod_{l\in L} G_{\mathrm{F}}^{r,\epsilon}(x_{fl}-x_{il})$$

be a regularization of an expression in perturbation theory. $l\in L$ is a line from a Feynman diagram, $r$ a regulator and $\epsilon$ the Feynman $\epsilon$-prescription. Then, there exists an $N$ for which

$$\lim_{\epsilon\to 0}\lim_{r\to 0}\prod_{l\in L} G_{\mathrm{F}}^{r,\epsilon}(x_{fl}-x_{il}) = \overline{F}(x_1,\dots,x_n) \tag{15.2}$$

where $\bar{F}(x_1,\dots,x_n)$ is a continuous functional over test functions $\rho_N$. Invoking the Hahn-Banach theorem for continous functionals on subspaces of a topological vector space $S$, which is satisfied due to our definition above, we can expand $\overline{F}$ from $S_N$ to $S$ in a continous way. This way, renormalization is just constructing this expansion mentioned in the Hahn-Banach theorem. This construction is *not* unique, however: one has to impose normalization conditions.

A theory is *renormalizable* if a finite set of conditions produces uniqueness to all orders of perturbation theory. The limit

$$\lim_{n\to\infty} (G^{\text{reg}}_{n,\text{F}}(x))^k$$

does not exist, but

$$F(x) = \lim_{n\to\infty} \left( (G^{\text{reg}}_{n,\text{F}}(x))^k + \Lambda^{\text{reg}}_n(x) \right) \qquad \text{with}$$
$$\Lambda \approx P(\partial)\delta(x) \text{ for } n \to \infty$$

is finite. This is the basis of the *Epstein-Glaser* regularization and renormalization scheme.

### 15.1.2 Pauli-Villars regularization

In the Pauli-Villars regularization scheme, one modifies $G_\text{F}$ by subtracting at some large, but finite mass $M$:

$$iG_\text{F}(x,m) \to iG^{\text{reg}}_\text{F} = i(G_\text{F}(x,m) - G_\text{F}(x,M)) \tag{15.3}$$

Writing out $G^{\text{reg}}_\text{F}$ gives:

$$\frac{1}{4\pi}\delta(x^2) - \frac{i}{2\pi^2}\frac{1}{x^2} - \frac{m^2}{16\pi^2}\theta(x^2) + \frac{im^2}{16\pi^2}\log\frac{m^2x^2}{4} + \mathcal{O}(\sqrt{x^2}\log\sqrt{x^2}) \to$$
$$\frac{M^2-m^2}{16\pi^2}\theta(x^2) + \frac{i}{16\pi^2}\left(m^2\log\frac{m^2x^2}{4} - M^2\log\frac{M^2x^2}{4}\right) + F(x,m,M)$$

with $F(x,m,M)$ a regular function. Note that we have used an explicit formula for $G_\text{F}$ here; we will just assume this identity, and skip the derivation. The $\delta(x^2)$ and $x^{-2}$, which gave singularities, have been cancelled.

One can take this idea further, and subtract a second term, at $M_2$:

$$G^{\text{reg}}_\text{F}(x,m,M_1,M_2) = G_\text{F}(x,m) - c_1G_\text{F}(x,M_1) - c_2G_\text{F}(x,M_2) \tag{15.4}$$

Taking these $c$'s to satisfy

$$1 - c_1 - c_2 = 0 \tag{15.5}$$

ensures that the $\delta(x^2)$ and $x^{-2}$ are still cancelled; we can add the constraint that

$$m^2 - c_1M_1^2 - c_2M_2^2 = 0 \tag{15.6}$$

In this case, also the $\theta(x^2)$- and $\log x^2$-terms are cancelled. Naturally, in the limit $M_i \to \infty$, we recover the old $G_{\mathrm{F}}(x,m)$. The auxiliary fields with the $M_i$ are indeed auxiliary, and unphysical, since the $c_i < 0$.

In momentum space, it looks like this:

$$\frac{i}{p^2 - m^2 + i\epsilon} \to \frac{i}{p^2 - m^2 + i\epsilon} - \frac{i}{p^2 - M^2 + i\epsilon} = \frac{i(m^2 - M^2)}{(p^2 - m^2 + i\epsilon)(p^2 - m^2 + i\epsilon)} \tag{15.7}$$

### 15.1.3 Cut-off regularization

We have seen this before; in its primitive form, it consists of simply introducing an integration boundary:

$$\int d^4 l_{\mathrm{E}} \to \int_{|l|<L} d^4 l_{\mathrm{E}}$$

However, the choice of $l$ in higher loop orders is very ambiguous, so this is not a very popular method.

In analytical cut-off regularization, one rewrites the propagator:

$$\frac{1}{p^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(p^2+m^2)} \to \int_{r>0}^\infty d\alpha e^{-\alpha(p^2+m^2)} \tag{15.8}$$

which is exponentially damped for large $p^2$. The parameter $\alpha$ which is introduced here is the so-called *Schwinger proper time*, used in world line quantization methods.

### 15.1.4 Dimensional regularization

This is the most popular regularization technique. It is almost exclusively used in gauge theories, since it is the only one that preserves the symmetries of all such theories.

The essential point is that one calculates the Feynman graph expression in a dimension different from 4, such that the integrals are finite there. Then, one goes back to $D = 4$, and, of course, finds back the old divergences and accompanying need for counterterms. This can be done without specifying the actual dimension one calculates in; even without specifying that it is an integral number. Put simply, one makes the substitution

$$\int \frac{d^4 k}{(2\pi)^4} \to \int \frac{d^D k}{(2\pi)^D} \tag{15.9}$$

with $D \neq 4$ and *unchanged* integrand.

Before we can work out what this means, we have to do some dimension-counting. The propagator in momentum space, for whichever dimension larger than two,

$$\int d^D x \, \langle 0| \, \mathrm{T}(\Phi(x)\Phi(0)) \, |0\rangle \, e^{ikx}$$

always has dimension $[p]^{-2}$, or 'momentum dimension -2' (we will be counting momentum dimensions here). We can see this from the fact that $\mathcal{L} = \partial\Phi\partial\Phi$ has dimension $D$, since $L$ is dimensionless, and that therefore $\Phi$ has dimension $(D-2)/2$. The dimension of the propagator can also be derived from this, or simply by looking at its Fourier transformed shape:

$$\frac{i}{p^2 - m^2 + i\epsilon}$$

A little more notation is needed: we will use

$$2\omega = D = 4 - 2\epsilon \tag{15.10}$$

assuming that we are calculating in some dimension close to 4.

We can still apply Feynman's formula for propagator products; this leaves us with the following integral to calculate:

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{i}{(k^2 + M^2 + 2kp)^A} \tag{15.11}$$

Completing the square rids us of the $2kp$-term in the denominator. The usual trick of converting the $N$-dimensional integral into one over the $N$-dimensional solid angle and one over $k^{N-1}$ works here as well:

$$\begin{aligned} &d^N k = k^{N-1} dk d\Omega_N \\ &d\Omega_N = d\phi \sin\theta_1 d\theta_1 \sin^2\theta_2 d\theta_2 \ldots \sin^{N-2}\theta_{N-2} d\theta_{N-1} \\ &\text{with} \quad 0 \le \phi \le 2\pi \quad \text{and} \quad 0 \le \theta_i \le \pi \end{aligned}$$

Performing the angular integration gives

$$\int d^N k = \int_0^\infty (k^2)^{(N-2)/2} dk^2 \frac{\pi^{N/2}}{\Gamma(N/2)}$$

The $k^2$-integration can be performed with the help of two variable substitutions: $x = k^2$ and $y = a^{-2}x$.

$$\int_0^\infty dx \frac{x^{(N-2)/2}}{(x + a^2)^A} = (a^2)^{-A+N/2} \int_0^\infty dy \, y^{(N-2)/2} \frac{1}{(1+y)^A}$$

Leaving out the prefactor, this integral reduces to the Beta-function, or a product of gamma functions:

$$\int_0^\infty dy\, y^{N/2-1}(1+y)^{-A} = B(N/2, A-N/2) = \frac{\Gamma(N/2)\Gamma(A-N/2)}{\Gamma(A)}$$

Setting $N = 2\omega$ then gives

$$\boxed{\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+M^2+2kp)^A} = \frac{\Gamma(A-\omega)}{(4\pi)^\omega \Gamma(A)} \frac{1}{(M^2-p^2)^{A-\omega}}} \tag{15.12}$$

**Action in $\Phi^4$-theory**

In $\Phi^4$-theory, the interaction contribution to the action is

$$S_\omega(\Phi) = \int d^{2\omega}x \left( \cdots - \frac{\lambda}{4!}(\mu^2)^{2-\omega}\Phi^4 \right)$$

The interaction term has to be multiplied with a quantity of nonzero dimension to keep the action itself dimensionless, since $\Phi$ has dimension $\omega - 1$, as was derived above. $\mu$ has momentum dimension 1 in the formula above, as can be derived by simply counting the dimensions of the various factors in the integrand: $2(2-\omega) + 4(\omega - 1) = 2\omega$.

**Gamma function**

The gamma function used above is the usual one, defined by

$$\begin{aligned} &N\Gamma(N) = \Gamma(N+1), \qquad \Gamma(N) = (N-1)! \qquad \text{or} \\ &\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1} \qquad \text{for complex } z \text{ with } \Re(z) > 0 \end{aligned}$$

Splitting up the integral from the last definition, we can rewrite it as

$$\begin{aligned} \Gamma(z) = \int_0^\alpha dt e^{-t} t^{z-1} + \int_\alpha^\infty dt e^{-t} t^{z-1} = \\ \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\alpha^{n+z}}{z+n} + \int_\alpha^\infty dt e^{-t} t^{z-1} \end{aligned}$$

We will also need $\Gamma(-n+\epsilon)$:

$$\begin{aligned} \Gamma(-n+\epsilon) = \frac{\Gamma(-n+1+\epsilon)}{-n+\epsilon} = \frac{\Gamma(\epsilon)}{(-n+\epsilon)\dots(-1+\epsilon)} = \frac{1}{\epsilon}\frac{\Gamma(1+\epsilon)}{(-n+\epsilon)\dots(-1+\epsilon)} = \\ \frac{(-1)^n}{n!}\left\{\frac{1}{\epsilon} + \psi(n+1) + \frac{1}{2}\epsilon\left(\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1)\right) + \mathcal{O}(\epsilon^2)\right\} \end{aligned}$$

with

$$\psi(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma$$
$$\psi'(n+1) = \frac{\pi^2}{6} + \sum_{k=1}^{n} \frac{1}{k^2}$$

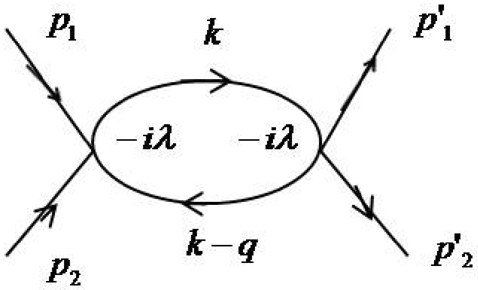
so

$$\psi'(1) = \frac{\pi^2}{6} \qquad \text{and}$$
$$\psi(1) = -\gamma = -0.5772$$

This $\gamma$ is the *Euler-Mascheroni constant*, which is known from number theory.

### 15.1.5 Simple example of dimensional regularization

Let us calculate a simple graph by means of dimensional regularization. We will take the following example, from $\Phi^4$-theory:




This graph has the following expression:

$$\frac{1}{2}\frac{\mu^{4-2\omega}}{(2\pi)^{2\omega}}(-i)^2\lambda^2 \int d^{2\omega}k \frac{i}{(k^2 - m^2 + i\epsilon)} \frac{i}{((k-q)^2 - m^2 + i\epsilon)}$$

Following the recipe from last chapter, we get

$$\frac{1}{2}\lambda^2\mu^{4-2\omega} i \int_0^1 d\alpha \int \frac{d^{2\omega}k_\mathrm{E}}{(2\pi)^{2\omega}} \left[\alpha(k_\mathrm{E}^2 + m^2) + (1-\alpha)((k-q)_\mathrm{E}^2 + m^2)\right]^{-2}$$

The inner integral can be rewritten with the help of eq. (15.12):

$$\int \frac{d^{2\omega}k'_\mathrm{E}}{(2\pi)^{2\omega}} \left[k_\mathrm{E}^2{}' + \alpha(1-\alpha)((k-q)_\mathrm{E}^2 + m^2) + m^2\right]^{-2} =$$
$$\frac{\Gamma(2-\omega)}{(4\pi)^\omega \Gamma(2)} \left[m^2 + \alpha(1-\alpha)q_\mathrm{E}^2\right]^{\omega-2}$$

Multiplying with $(4\pi)^2$ (we will divide this out again, at the end), we can rewrite the integrand of the $\alpha$-integral by means of the following trivial identity:

$$\left[\frac{4\pi\mu^2}{m^2+q_{\mathrm{E}}^2\alpha(1-\alpha)}\right]^{2-\omega} = \exp\left\{\log\left[\frac{4\pi\mu^2}{m^2+q_{\mathrm{E}}^2\alpha(1-\alpha)}\right]^{\epsilon}\right\} \approx$$
$$1+\epsilon\log\left[\frac{4\pi\mu^2}{m^2+q_{\mathrm{E}}^2\alpha(1-\alpha)}\right]$$

Returning to the $\alpha$-integral, defining $a := 4m^2/q_{\mathrm{E}}^2$ and only considering the denominator of the argument of log, this gives

$$\int_0^1 d\alpha \log\left\{1+\alpha(1-\alpha)\frac{4}{a}\right\} = -2+\sqrt{1+a}\log\left\{\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right\}$$

All of this gives the following final result:

$$\frac{1}{2}\frac{\mu^{4-2\omega}}{(2\pi)^{2\omega}}(-i)^2\lambda^2\int d^{2\omega}k\frac{i}{(k^2-m^2+i\epsilon)}\frac{i}{((k-q)^2-m^2+i\epsilon)} =$$
$$i\lambda^2\frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+\psi(1)-\frac{\epsilon}{\epsilon}(-2)+\frac{\epsilon}{\epsilon}\log\frac{4\pi\mu^2}{m^2}-\right.$$
$$\left.\frac{\epsilon}{\epsilon}\sqrt{1+\frac{4m^2}{q_{\mathrm{E}}^2}}\log\left\{\frac{(1+4m^2/q_{\mathrm{E}}^2)^{1/2}+1}{(1+4m^2/q_{\mathrm{E}}^2)^{1/2}-1}\right\}+\mathcal{O}(\epsilon)\right)$$

## 15.2 Various renormalizations

We have already seen physical renormalization, which consisted of bringing the propagator into the standard form $(i/(p^2-m^2+i\epsilon))$ around $p^2=m_{\mathrm{p}}^2$. Another option is *intermediate renormalization*; this is similar to the physical variant, except that one brings the propagator into the standard form around $p^2=0$. This brings along the following conditions:

$$\begin{aligned}
\tau(p,-p)^{-1}\Big|_{p^2=0} &= \frac{-m^2}{i}\\
\left.\frac{d\tau(p,-p)^{-1}}{dp^2}\right|_{p^2=0} &= \frac{1}{i}\\
\Lambda\big|_{p_i=0} &= i\lambda_{\mathrm{ren}}
\end{aligned}$$

(see chapter 14 for the definition of $\lambda_{\mathrm{ren}}$). Another option is to renormalize at $p^2=-\mu^2$; the accompanying renormalization conditions are determined in analogy to the case above. This renormalization avoids the infrared singularities that arise for $m^2=0$, e.g. for the gauge fields in gauge theories.

The most important one is renormalization in the framework of dimensional regularization. One subtracts the $\epsilon^{-1}$-singularities; this is called *minimal subtraction*, also denoted by *MIN*. Since these singularities are always accompanied by the terms $\log 4\pi$ and $\Psi(1)$, these are often subtracted as well; this is denoted by *MIN′*, and called *minimal prime subtraction*.

The masses and coupling constants are defined differently in the various renormalization schemes; e.g., $m(\mu) \neq m_{\rm p}$ for $\mu^2$-renormalization. As a consequence, physically identical $n$-point functions are expressed differently in terms of masses and couplings. These differences are in the finite parts of the counterterms.

Renormalization theory states that for renormalizable theories, the number of counterterms is finite. These counterterms are power series in the coupling constant $\lambda$; the coefficients of these series are fixed in perturbation theory. For example, in $\Phi^4$-theory, we have

$$\begin{aligned}
\Phi_{\rm ren} &= \frac{1}{\sqrt{Z_2}}\Phi_0, \qquad & Z_2 &= 1 + \ldots\lambda^2 + \ldots\lambda^3 + \ldots \\
& & \delta m^2 &= \ldots\lambda m^2 + \ldots\lambda^2 m^2 + \ldots \\
\lambda_{\rm ren} &= \frac{Z_2^2}{Z_1}\lambda_0, \qquad & Z_1 &= 1 + \ldots\lambda^2 + \ldots
\end{aligned}$$

# Chapter 16

# The renormalization program in higher orders

In this chapter, we will discuss the renormalization program in general terms. This might be of little interest to those who simply want to calculate a graph to a given (low) loop order, but is necessary for a complete treatment of quantum field theory.

## 16.1 Superficial divergence

By means of a procedure called 'Dyson counting', it is possible to make a statement about the *superficial degree of divergence* of a graph. What this superficial degree of divergence is will become clear in the course of this section.

Consider a general graph. Let us call the number of outer points $A$, the number of outer and inner propagators $P$, the number of loops $L$ and the number of vertices $V$. Then we have, in our well-known $\Phi^4$-theory,

$$\boxed{4V + A = 2P} \tag{16.1}$$

since all propagators have two endpoints, and vertices are endpoints for four propagators each. Cutting a line that is part of a loop (see fig. 16.1) results in:

$$\begin{aligned} \Delta A &= 2 \\ \Delta P &= 1 \\ \Delta L &= -1 \end{aligned}$$

Naturally, after cutting $L$ such lines, we have eliminated all loops, and are left with a tree level graph (to be denoted by a subscript '0'). For tree level graphs, we know

$$P_0 - A_0 = V_0 - 1$$

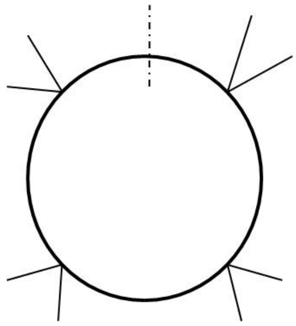


Figure 16.1: Cutting an inner line

(this can be checked by induction: begin with a single propagator and add vertices). From the cutting procedure, we also know

$$
\begin{aligned}
A_0 &= A + 2L \\
P_0 &= P + L \\
V_0 &= V
\end{aligned}
$$

From all of this, we can conclude

$$\boxed{P - A - L = V - 1} \tag{16.2}$$

Plugging in eq. (16.1) into eq. (16.2), we come to

$$L = \frac{P}{2} - \frac{3}{4}A + 1 \tag{16.3}$$

Now, considering the general form of a Feynman loop integral, we define the superficial degree of divergence $\omega(\Gamma)$ of a graph $\Gamma$:

$$\boxed{\omega(\Gamma) = 4L - 2(P - A)} \tag{16.4}$$

Using eq. (16.3), we can rewrite this to

$$\omega = 2P - 3A + 4 - 2P + 2A = 4 - A$$

This means that $n$-point functions with $A > 4$ are *superficially convergent*. Now let us qualify what 'superficially' means. The fact that $\omega$ indicates convergence can easily be checked for 1-loop amplitudes by power counting: loops give dimension 4 due to the integral, propagators dimension -2, and outer points 2 again (we are still counting momentum dimensions, like in chapter 15). In higher loop order graphs, however, divergent subgraphs may be present: see fig. 16.2, where the left graph has $\omega(\Gamma) = -2$, but the right graph, which is a subgraph of the left one, has $\omega(\Gamma) = 0$.

So, even if we have $\omega(\Gamma) < 0$ for a graph, we still have to decompose it into all possible subgraphs to see if they are not divergent.

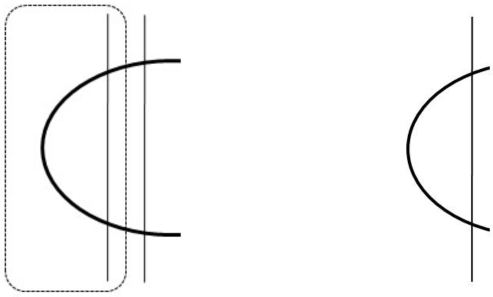


Figure 16.2: Superficially convergent graph (left, $\omega(\Gamma) = -2$) with divergent subgraph (right, $\omega(\Gamma) = 0$)

## 16.2 Graphology

For this decomposition, we have to consider only the truncated versions of $n$-point functions, where outer lines, including self-energy contributions, have been removed:

$$G^{(n)}_{\text{trunc}}(p_1, \dots, p_n) = \prod_{k=1}^{n} \left[G^{(2)}(p_k, -p_k)\right]^{-1} G^{(n)}(p_1, \dots, p_n) \tag{16.5}$$

Remember that for unrenormalized fields, using $G^{(2)}(p, -p) = (iZ)^{-1}(p^2 - m^2)$, we have for $p^2 \approx m^2$

$$\begin{aligned}\langle p_1 \dots p_n \,\text{out}| q_1 \dots q_m \,\text{in}\rangle =& Z^{(n+m)/2} G^{(n+m)}_{\text{trunc}}(-p_1, \dots, -p_n, q_1, \dots, q_m) \times \\ & (2\pi)^4 \delta^4 \left(\sum p_i - \sum q_i\right)\end{aligned}$$

with $p_i^2 = q_i^2 = m^2$. For renormalized fields, the $Z$-factors drop out, since they only appear in counterterms which subtract divergent graph expressions.

We can also limit our investigation to connected 1PI diagrams, also called *proper diagrams*. These graphs contain subgraphs $\gamma \in \Gamma$; a subgraph is defined as the set of all lines connecting a set of vertices $\{V'_1, \dots, V'_m\} \subset \{V_1, \dots, V_n\}$ of a graph $\Gamma$ (in German, this is called a *Teilgraph*).

These subgraphs can have certain properties:

- Two subgraphs $\gamma_1$ and $\gamma_2$ are called *disjunct* if they have no common vertices or lines:

$$\gamma_1 \cap \gamma_2 = \emptyset$$

- A subgraph is called *connected* if it is not the sum of disjunct graphs.

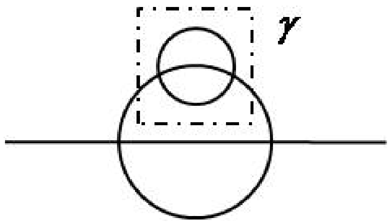



Figure 16.3: Subgraph

- Two subgraphs $\gamma_1$ and $\gamma_2$ are called *overlapping*, which is denoted by $\gamma_1 \circ \gamma_2$, if neither is completely contained in the other, but they *do* share at least one line or vertex; in other words, if the following statements are *not* true:

$$\gamma_1 \subset \gamma_2 \quad \text{or} \quad \gamma_2 \subset \gamma_1 \quad \text{or} \quad \gamma_1 \cap \gamma_2 = \emptyset$$

Decomposing graphs into superficially divergent subgraphs and treating these divergencies locally, i.e. in the subgraphs, yields so-called *skeleton graphs*:

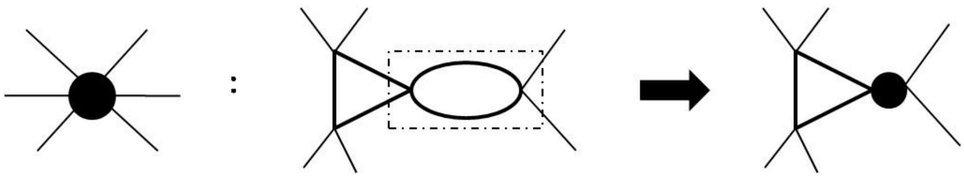


Figure 16.4: Skeleton graphs

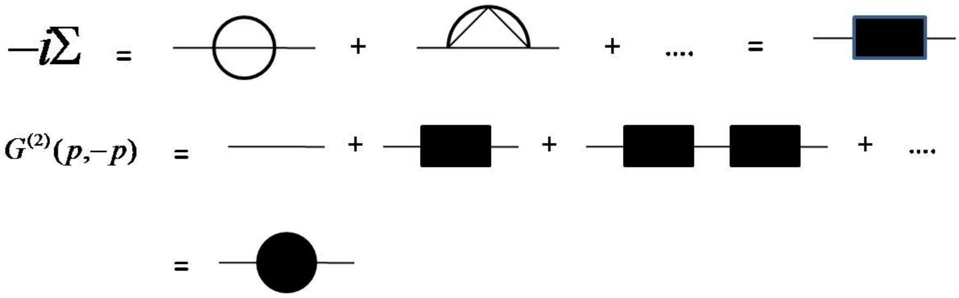



Figure 16.5: Self-energy and full propagator

The Dyson counting procedure is exact for these skeleton graphs, since the divergencies resulting from subgraphs have been dealt with. The subgraphs, which are superficially and actually divergent (e.g. 2- and 4-point functions in $\Phi^4$-theory), are called *renormalization pieces.*

**Weinberg's theorem** says that a Feynman diagram corresponds to a convergent expression if the sum of divergence degrees of the graph itself and all its subgraphs is negative.

### Renormalization pieces

In $\Phi^4$-theory, as mentioned, the renormalization pieces are:

(i) the self-energy $\Sigma$: the sum of all 1PI truncated 2-point functions (fig. 16.5)

(ii) the vertex function $\Lambda^4$ (fig. 16.6)

$$\Lambda^4(p_1,\ldots,p_4) = \sum_\Gamma G_\Gamma^{(4)}(p_1,\ldots,p_4)$$

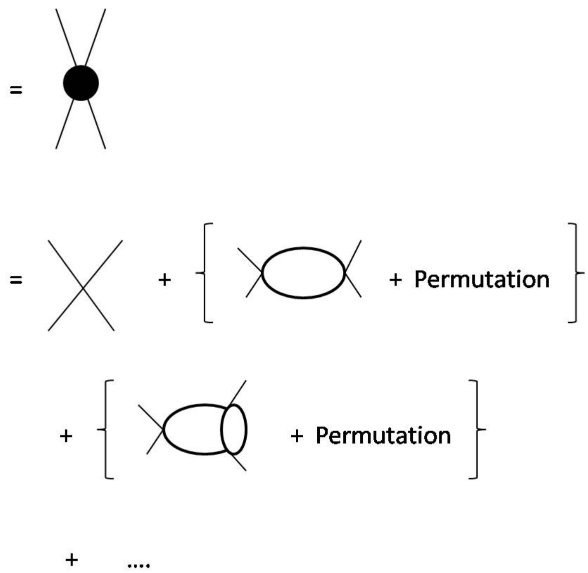
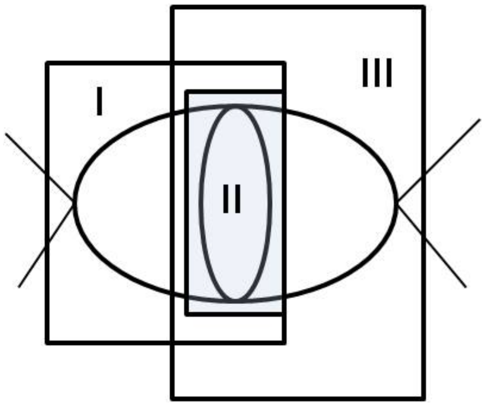



Figure 16.6: Vertex function

The renormalization pieces are regularized and made finite by stepwise adding counterterms in each divergent subgraph. These counterterms will be contained in the new set of Feynman rules, as we have seen before. In proofs of renormalizability, it is better to avoid writing out the counterterms explicitly, and introduce the renormalization pieces in already subtracted form instead. In doing this, it appears to be a problem that the decomposition into skeleton graphs with renormalization pieces is not unique, i.e. one gets "overlapping divergencies": see fig. 16.7.




captionOverlapping divergent subgraphs

Of course, these problems can be avoided by simply calculating, with e.g. dimensional regularization, taking into account the counterterms in the Feynman rules.

**Integral equations**

For both $\Sigma$ and $\Lambda$, we need to consider 'nested' situations: ultimately, the self-energy is given by the Schwinger-Dyson equations, or the following graph:

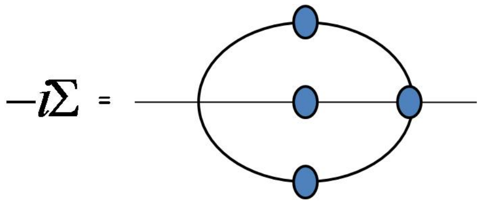



Figure 16.7: the full propagator

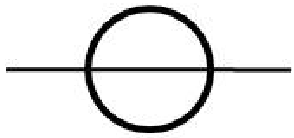
For $\Lambda$, the expression is similar, but more complicated.

## 16.3 Direct regularization and renormalization

The proof of renormalizability due to Bogoliubov, Parasiuk, Hepp and Zimmermann ("BPHZ") will be indicated here. It is based on intermediate renormalization:

$$\Lambda^{(4)}_{p_i=0} = i\lambda$$
$$\Gamma^{(2)}\Big|_{p^2=0} = im^2; \qquad \frac{\partial}{\partial p^2}\Gamma^{(2)}\Big|_{p^2=0} = -i$$

Consider the Taylor series of the divergent integrand. The first, divergent, terms are the negatives of the counterterms (this defines the regulator), and are hence canceled. For example, for the graph



**Γ**

one obtains the following expression:

$$\left(1 - t^{\omega(\Gamma)}\right)\left[\int \ldots\right]^{\mathrm{reg}}$$

$T_\Gamma := t^{\omega(\Gamma)}$ is defined in Fourier space as the Taylor expansion in outer momentum $p$ to order $\omega(\Gamma)$, and it acts in the *integrand*. Now, let $\bar{\mathcal{R}}_G$ be a finite integrand, i.e. with the divergencies from subgraph integrands already removed. Then,

$$\mathcal{R}_G = (1 - T_G)\bar{\mathcal{R}}_G \tag{16.6}$$

for $\omega(G) \geq 0$, i.e. if the integral is divergent; otherwise, $\mathcal{R}_G = \bar{\mathcal{R}}_G$. Note that $\bar{\mathcal{R}}_G$ is in general *not* the naïve Feynman expression (which will be called $J_G$), but already contains counterterms, in order for it to have only finite subgraphs, or renormalization pieces, $\gamma \subset G$. The contribution of one such counterterm is then given by

$$J_{G/\gamma}(-T_\gamma \bar{\mathcal{R}}_\gamma)$$

Two such counterterms would result in a contribution

$$J_{G/\{\gamma_1,\gamma_2\}}(-T_{\gamma_1}\bar{\mathcal{R}}_{\gamma_1})(-T_{\gamma_2}\bar{\mathcal{R}}_{\gamma_2}) \quad \text{for} \quad \gamma_1 \cap \gamma_2 = \emptyset$$

All in all, we obtain $(1 - T_{\gamma_1})(1 - T_{\gamma_2})\bar{\mathcal{R}}_G$. In general, one has

$$\boxed{\bar{\mathcal{R}}_G = J_G + \sum_{\{\gamma_1,\ldots,\gamma_n\},\gamma_i\cap\gamma_j=\emptyset} J_{G/\{\gamma_1,\ldots,\gamma_n\}} \prod_{a=1}^{n}(-T_{\gamma_a}\bar{\mathcal{R}}_{\gamma_a})} \tag{16.7}$$

where the sum is over all combinations of disjunct subgraphs, including the individual $\gamma_i$. This formula can be iterated for the $\bar{\mathcal{R}}_{\gamma_a}$, resulting in a nested set of graphs. Note that different summands contain different nested systems, and that all are summed; this gives the right prescription for overlapping divergencies, and thus solves the problem mentioned above. For an example, see Itzykson & Zuber, p. 392 ($\Phi^3$ in 6 dimensions).

### Explicit solution

From the above, an explicit solution can be derived, which bears Zimmermann's name. It is based on a set of nested, non-overlapping renormalization pieces, represented by Van diagrams, as in fig. 16.9.

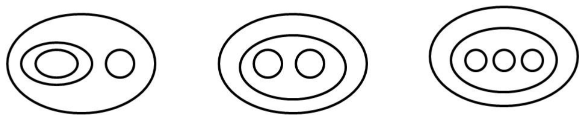


Figure 16.8: Van diagrams

These can also be represented as "trees", forming a "forest", in the following way:

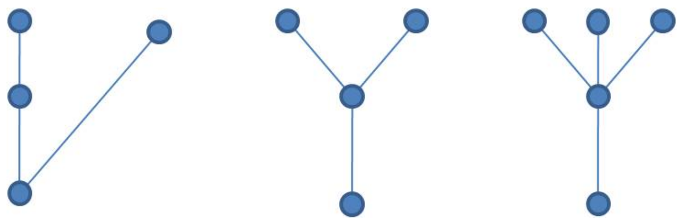


Figure 16.9: The forest

The explicit solution after iteration is then

$$\mathcal{R}_G = \sum_U \prod_{\gamma \in U} (-T_\gamma) J_G \tag{16.8}$$

where $U$ is the forest. This gives a finite Feynman integral (after summing and integrating). In principle, no regulator is needed, although it can still be useful for inspecting the various contributions.

### Note

One can also consider *partial graphs* (*Untergraphen* in German), parts of graphs that are not necessarily subgraphs as defined above, i.e. do not need to contain all lines connecting the points in the subgraph. Doing a Taylor expansion for all divergent partial graphs will lead to an analogous formula.

# Chapter 17

# Effective action

## 17.1 Effective action in analogy to statistical mechanics

We have already encountered the generating functional

$$Z(j) = \int \mathcal{D}\Phi \exp\left\{-S(\Phi) - \int d^4x j(x)\Phi(x)\right\}$$

with the action

$$S(\Phi) = \int d^4x \left(\frac{1}{2}\partial_\mu\Phi\partial_\mu\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi)\right)$$

(both written in their Wick-rotated, i.e. Euclidean, versions). $Z$ is normalized to

$$Z(0) = 1$$

(if it is not, it can always be renormalized).

#### Spin in magnetic field

Note the similarity between the generating functional and the *partition function* from statistical mechanics; consider the example of a spin system in a magnetic field $H$:

$$Z(H) = \int \mathcal{D}s \exp\left\{-\beta \int dx(\mathcal{H}(s) - Hs(x))\right\} \tag{17.1}$$

where $\mathcal{H}(s)$ is the spin energy density and $\beta = (k_\mathrm{B}T)^{-1}$.

The *Helmholtz free energy* $F(H)$ is defined by

$$Z(H) = e^{-\beta F(H)} \qquad \text{or} \qquad F(H) = -kT\log Z \tag{17.2}$$

The magnetization $M$ of the system is obtained by differentiating the free energy:

$$\begin{aligned} -\frac{\partial F}{\partial H}\bigg|_{\beta\,\text{fixed}} &= \frac{1}{\beta}\frac{\partial}{\partial H}\log Z = \\ & \frac{1}{Z}\int dx \int \mathcal{D}s\, s(x)\exp\left\{-\beta\int dx(\mathcal{H}(s) - Hs(x))\right\} = \\ & \int dx\langle s(x)\rangle = M \end{aligned} \tag{17.3}$$

The *Gibbs free energy* is defined as the *Legendre transform* of $F$ given by

$$G = F + MH \tag{17.4}$$

where $G = G(M)$, i.e. $G$ depends on the derivative of $F$ with respect to $H$. Differentiating $G$ (still keeping $\beta$ fixed) gives

$$\begin{aligned} \frac{\partial G}{\partial M} &= \frac{\partial F}{\partial M} + M\frac{\partial H}{\partial M} + H = \\ & \frac{\partial F}{\partial H}\frac{\partial H}{\partial M} + M\frac{\partial H}{\partial M} + H = H \quad \text{for fixed } \beta \end{aligned}$$

So, at $H = 0$, $\partial G/\partial M = 0$, i.e., $G$ is at an extremum.

**Generating functional**

Similar to the relation between partition function and Helmholtz free energy, one can write for the generating functional

$$Z(j) = e^{-W(j)} = \int \mathcal{D}\Phi \exp\left\{-\int d^4x(\mathcal{L}(\Phi) + j\Phi)\right\}$$

where $W(j)$ is the general vacuum energy in the presence of a source $j(x)$. Following the analogy, we find

$$\boxed{\frac{\delta W(j)}{\delta j(x)} = -\frac{1}{Z}\frac{\delta Z}{\delta j(x)} = \langle\Phi(x)\rangle_j}$$

We can also obtain correlation functions this way:

$$\langle\Phi(x_1)\dots\Phi(x_n)\rangle_{j=0} = (-1)^n \left.\frac{\delta^n Z}{\delta j(x_1)\dots\delta j(x_n)}\right|_{j=0} \tag{17.5}$$

(The usual factor of $1/Z(0)$ has been omitted here, since we assume $Z(0) = 1$.) The $\Phi$ are field operators in the canonical formalism. We can use this

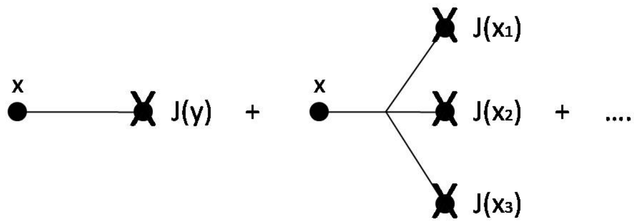



Figure 17.1: Solving for $\Phi(x)$ iteratively

result to rewrite $Z(j)$ and $W(j)$:

$$Z(j) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int dx_1 \dots dx_n \langle \Phi(x_1) \dots \Phi(x_n) \rangle_{j=0} j(x_1) \dots j(x_n) \quad (17.6)$$

$$W(j) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \int dx_1 \dots dx_n \langle \Phi(x_1) \dots \Phi(x_n) \rangle_{j=0,\text{conn.}} j(x_1) \dots j(x_n)$$

$W(j)$ thus becomes the generator of the *connected* Green's functions. To check this, consider the following examples, for $n = 2$ and $n = 3$:

$$\frac{\delta^2 W}{\delta j(x_1) \delta j(x_2)} = \langle \Phi(x_1)\Phi(x_2) \rangle - \langle \Phi(x_1) \rangle \langle \Phi(x_2) \rangle = \langle \Phi(x_1)\Phi(x_2) \rangle_{\text{c}}$$

$$\begin{aligned} \langle \Phi(x_1)\Phi(x_2)\Phi(x_3) \rangle_{\text{c}} = & \langle \Phi(x_1)\Phi(x_2)\Phi(x_3) \rangle - \langle \Phi(x_1) \rangle \langle \Phi(x_2) \rangle \langle \Phi(x_3) \rangle - \\ & \langle \Phi(x_1)\Phi(x_2) \rangle_{\text{c}} \langle \Phi(x_3) \rangle - \langle \Phi(x_2)\Phi(x_3) \rangle_{\text{c}} \langle \Phi(x_1) \rangle - \\ & \langle \Phi(x_3)\Phi(x_1) \rangle_{\text{c}} \langle \Phi(x_2) \rangle \end{aligned}$$

This expansion of $W(j)$ can be interpreted as follows: from the classical equation of motion for $\Phi(x)$, we have

$$\begin{aligned} (-\partial^2 + m^2)\Phi(x) + \frac{\lambda}{3!}\Phi^3(x) + j(x) = 0 \qquad \text{so} \\ \Phi(x) = -\int dy G(x,y) \left( \frac{\lambda}{3!}\Phi^3(y) + j(y) \right) \end{aligned}$$

We can solve for $\Phi(x)$ by iterating this formula, which in graphs looks like this:

Functional differentiation of the fourth term of $W(j)_{\text{class.}}$ gives

$$\frac{\lambda}{4!} \int dy \, dx_1 dx_2 dx_3 dx_4 \, j(x_1) \dots j(x_4) G(y,x_1) \dots G(y,x_4)$$

So, $W(j)_\mathrm{c}$ is the sum of all tree level graphs with $j$'s at the outer points and $\Phi(x) = \delta W/\delta j(x)$. The full, quantum $W(j)$, however, also contains loops.

Now, let us, in analogy to the statistical mechanics example discussed above, apply a Legendre transform:

$$\boxed{j(x) \leftrightarrow \frac{\delta W}{\delta j(x)} = \langle \Phi(x) \rangle_j = \varphi} \tag{17.7}$$

The analogue to the Gibbs free energy is the *effective action*:

$$\Gamma(\varphi) = W(j) - \int d^4x \varphi(x) j(x) \tag{17.8}$$

Differentiating this with respect to our new variable $\varphi$ (this time, it's a functional derivative) gives

$$\begin{aligned}
\frac{\delta \Gamma}{\delta \varphi(y)} = \frac{\delta W}{\delta \varphi(y)} - \int d^4x \frac{\delta \varphi(x)}{\delta \varphi(y)} j(x) - \int d^4x \varphi(x) \frac{\delta j(x)}{\delta \varphi(y)} =& \\
\int d^4x \frac{\delta W(j)}{j(x)} \frac{\delta j(x)}{\delta \varphi(y)} - \int d^4x \, \delta^4(x-y) j(x) - \int d^4x \varphi(x) \frac{\delta j(x)}{\delta \varphi(y)} =& \\
\int d^4x \varphi(x) \frac{\delta j(x)}{\delta \varphi(y)} - j(y) - \int d^4x \varphi(x) \frac{\delta j(x)}{\delta \varphi(y)} =& \\
-j(y)&
\end{aligned}$$

If we switch off the source, we see that $\Gamma$ is at an extremum:

$$\frac{\delta \Gamma}{\delta \varphi(y)} = 0$$

Note that this constitutes a generalization of the classical action, for which we had

$$\frac{\delta S}{\delta \Phi(y)} = -j(y)$$

If $j$ is a constant we obtain the *effective potential* $V$ for a constant field $\varphi$:

$$\begin{aligned}
\Gamma(\varphi) \to V(\varphi) &= \text{volume} \times \mathcal{V}(\varphi) \qquad \text{and} \\
\frac{\partial \mathcal{V}}{\partial \varphi} + j &= 0
\end{aligned} \tag{17.9}$$

## Note

If $\mathcal{V}(\varphi)$ is not convex, there will be degenerate minima, which allow for a quantummechanical superposition of states. In statistical mechanics, this comes up in the discussion of first order phase transitions (see e.g. Maxwell construction or tangential construction); these can be discussed in the context of the Wilson renormalization group.

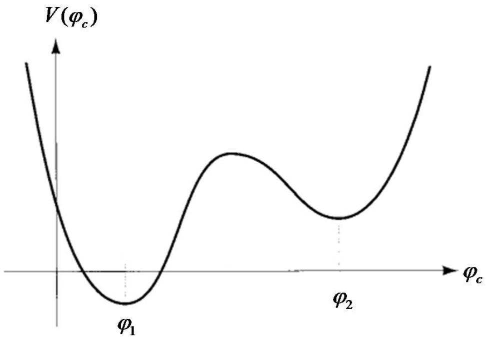



Figure 17.2: $\mathcal{V}$ is not convex

## 17.2 Computation of the effective action

$\Gamma(\varphi)$ corresponds to the sum of all 1PI graphs in a background of outer $\varphi$:

$$\Gamma(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_4 \Gamma^{(n)}(x_1, \dots, x_n)\varphi(x_1) \dots \varphi(x_n) \tag{17.10}$$

A hint on how to prove this: write $\Phi(x)$ as

$$\Phi(x) = \varphi(x) + \eta(x)$$

with $\langle \eta(x) \rangle = 0$ (adjust $j(x)$ to achieve this) and do perturbation theory in $\eta$:

$$\begin{aligned} &\frac{1}{2}m^2\,\Phi^2 + \frac{\lambda}{4!}\Phi^4 - j\Phi = \\ &\quad \frac{1}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4 - j\varphi + \eta\left(-j + m^2\varphi + \frac{\lambda}{3!}\varphi\right)^3 + \\ &\quad \frac{1}{2}\eta^2\left(m^2 + \frac{\lambda}{2}\varphi^2\right) + \frac{\lambda}{3!}\eta^3\varphi + \frac{\lambda}{4!}\eta^4 \end{aligned}$$

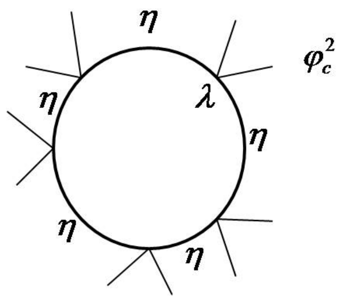
For 1PI diagrams in $W(j)$ (remember, these are the connected diagrams), this means

if the condition $\langle \eta(x) \rangle = 0$ is equivalent to the one that $j(x) + \Gamma_1(x) = 0$. Therefore, we have

$$\Gamma(\varphi) = W(j) - \int d^4x \varphi(x) j(x)$$

where the $j$-dependence cancels on the left-hand side.

The 1-loop 1PI graphs for the *effective potential* are of the type




and have in their corresponding expressions a path integral over $\eta$, with an action which is quadratic in $\eta$ ($\lambda\varphi^2/2$ is a contribution to $m^2$). Since they are connected diagrams, $W(0) = 0$ and $\Gamma(0) = 0$; all of this leads to the expression

$$-\log\det\left[\frac{(-\partial^2 + m^2 + \lambda\varphi^2/2)}{(-\partial^2 + m^2)}\right]^{-1/2} = \frac{1}{2}\log\det\left[\frac{(-\partial^2 + m^2 + \lambda\varphi^2/2)}{(-\partial^2 + m^2)}\right] =$$
$$\frac{1}{2}\log\det\left[\left(\frac{\delta^2 S}{\delta\Phi(x)\delta\Phi(y)}\right)\Big/\left(\frac{\delta^2 S_0}{\delta\Phi(x)\delta\Phi(y)}\right)\right]$$

We get the general formula

$$\boxed{\Gamma(\varphi) = S(\varphi) + \frac{1}{2}\log\det\left[\left(\frac{\delta^2 S}{\delta\Phi(x)\delta\Phi(y)}\right)\Big/\left(\frac{\delta^2 S_0}{\delta\Phi(x)\delta\Phi(y)}\right)\right] + \text{higher loop orders} \qquad (17.11)}$$

Using the identity $\log\det = \operatorname{tr}\log$ and writing out the trace explicitly for constant $\varphi$, the second term becomes

$$\frac{1}{2}\text{tr}\,\log\left[\frac{(-\partial^2+m^2+\lambda\varphi^2/2)}{(-\partial^2+m^2)}\right]=$$
$$\frac{1}{2}\int d^4x\left\langle x\left|\log\left(\frac{(-\partial^2+m^2+\lambda\varphi^2/2)}{(-\partial^2+m^2)}\right)\right|x\right\rangle=$$
$$\frac{1}{2}\int d^4x\int\frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\langle x|k\rangle\left\langle k\left|\log\left(\frac{(-\partial^2+m^2+\lambda\varphi^2/2)}{(-\partial^2+m^2)}\right)\right|k'\right\rangle\langle k'|x\rangle=$$
$$\frac{1}{2}\int d^4x\int\frac{d^4k}{(2\pi)^4}\log\left(\frac{k^2+m^2+\lambda\varphi^2/2}{k^2+m^2}\right)=$$
$$\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\log\left(\frac{k^2+m^2+\lambda\varphi^2/2}{k^2+m^2}\right)\times V_4$$

In the third step, we have used the fact that $\langle k|\log(\dots)|k'\rangle$ gives a factor of $(2\pi)^4\delta^4(k-k')$.

### Renormalization of the effective potential (1-loop) of $\Phi^4$

The effective potential, to first loop order, of $\Phi^4$, is

$$\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\log\left(k^2+m^2+\frac{\lambda}{2}\varphi^2\right)+\{V_{\text{tree}}\}+\{\text{counterterms}\}$$

Rewriting the logarithm, we can turn the first term into

$$-\frac{1}{2}\frac{\partial}{\partial\alpha}\int\frac{d^4k}{(2\pi)^4}\left.\frac{1}{(k^2+m^2+\lambda\varphi^2/2)^\alpha}\right|_{\alpha=0}$$

Going to $D$ dimensions, for dimensional regularization, we obtain

$$-\frac{1}{2}\frac{\partial}{\partial\alpha}\int\frac{d^Dk}{(2\pi)^D}\left.\frac{1}{(k^2+m^2+\lambda\varphi^2/2)^\alpha}\right|_{\alpha=0}=$$
$$-\frac{1}{2}\frac{\partial}{\partial\alpha}\left.\left(\frac{1}{(4\pi)^{D/2}}\frac{\Gamma(\alpha-D/2)}{\Gamma(\alpha)}\frac{1}{(m^2+\lambda\varphi^2/2)^{\alpha-D/2}}\right)\right|_{\alpha=0}=$$
$$-\frac{1}{2}\frac{\Gamma(-D/2)}{(4\pi)^{D/2}}\left(m^2+\frac{\lambda}{2}\varphi^2\right)^{D/2}$$

Plugging in $2\omega=D=4-2\epsilon$ gives

$$-\frac{1}{2}\frac{\Gamma(-2+\epsilon)}{(4\pi)^{2-\epsilon}}\left(m^2+\frac{\lambda}{2}\varphi^2\right)^{2-\epsilon}= \tag{17.12}$$
$$-\frac{1}{2}\left(\frac{(-1)^2}{2!}\left\{\frac{1}{\epsilon}+\Psi(2+1)+\dots\right\}e^{\epsilon\log 4\pi}e^{-\epsilon\log(m^2+\lambda\varphi^2/2)}\right)\frac{(m^2+\lambda\varphi^2/2)^2}{(4\pi)^2}=$$
$$-\frac{1}{4}\frac{(m^2+\lambda\varphi^2/2)^2}{(4\pi)^2}\left\{\frac{1}{\epsilon}+\Psi(3)+\log 4\pi-\log(m^2+\lambda\varphi^2/2)\right\}+\mathcal{O}(\epsilon)$$

With

$$V_{\text{tree}} = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4$$

we can impose the following (natural) normalization conditions:

$$V(\varphi = m/\sqrt{\lambda}) = V_{\text{tree}}(\varphi = m/\sqrt{\lambda}) = \frac{m^4}{\lambda}\left(\frac{1}{2} + \frac{1}{4!}\right) \tag{17.13}$$

and similarly

$$\frac{\partial V}{\partial \varphi^2}(\varphi = m/\sqrt{\lambda}) = \frac{1}{2}m^2 + \frac{2\lambda}{4!}\frac{m^2}{\lambda} = \frac{1}{2}m^2 + \frac{2m^2}{4!} \tag{17.14}$$

Adding the one-loop contribution (eq. (17.12)), including the counterterms $-\frac{1}{2}\delta m^2\varphi^2 - \frac{\delta\lambda}{4!}\varphi^4$, we get from eq. (17.13)

$$-\frac{1}{2}\delta m^2\frac{m^2}{\lambda} - \frac{\delta\lambda}{4!}\frac{m^4}{\lambda^2} - \frac{1}{4(4\pi)^2}\left(m^2 + \frac{m^2}{2}\right)^2 \times$$
$$\left\{\frac{1}{\epsilon} + \Psi(3) + \log 4\pi - \log\frac{3m^2}{2}\right\} = \frac{m^4}{\lambda}\left(\frac{1}{2} + \frac{1}{4!}\right)$$

and from eq. (17.14)

$$-\frac{1}{2}\delta m^2 - \frac{2\delta\lambda}{4!}\frac{m^2}{\lambda} - \frac{1}{2(4\pi)^2}\left(m^2 + \frac{m^2}{2}\right)\frac{\lambda}{2} \times$$
$$\left\{\frac{1}{\epsilon} + \Psi(3) + \log 4\pi - \log\frac{3m^2}{2}\right\} + \frac{1}{4}\frac{(3m^2/2)^2}{(3m^2/2)}\frac{\lambda}{2(4\pi)^2} = m^2\left(\frac{1}{2} + \frac{2m^2}{4!}\right)$$

which can be solved for $\delta m^2$ and $\delta\lambda$.

**Notes**

- There are various other normalization conditions being used in the literature, one example being renormalization at the minimum for spontaneously broken symmetry. If that is at $\varphi = 0$, we need conditions for $\partial V/\partial\varphi^2$ and $\partial^2 V/(\partial\varphi^2)^2$ at $\varphi = 0$.
- Also note that in principle, we should have subtracted $V(0)$ at the beginning, in order to obtain $V(0) = 0$.

## 17.3 Remark on perturbation theory

Having observed that the effective potential contains all orders of $\lambda$, consider the following. Let us study a toy model, (1+0)-dimensional "field theory" (actually, in one space and zero time dimensions, we are dealing with normal one-dimensional QM), with

$$Z(\lambda) = \int_{-\infty}^{\infty} dx e^{-x^2-\lambda x^4} = \int_{-\infty}^{\infty} dx \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} x^{4n} e^{-x^2} \tag{17.15}$$

Exchanging integration and summation and performing the integral, this turns into

$$Z(\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^n \frac{\Gamma(2n+1/2)}{n!}$$

Now, $\Gamma(2n+\frac{1}{2})/n! \approx 4^n n!$ in the Stirling approximation ($n! \approx \sqrt{2\pi n} e^{n \log n}$), which holds for large $n$, so it grows factorially, i.e. very fast. $Z(\lambda)$ is not uniformly convergent, since there is an (essential) singularity at $\lambda = 0$. If $\Re\lambda < 0$, the integral is divergent, which is related to the instability of the vacuum in $\Phi^4$-theory. It can be defined by analytical continuation, but there is a branch cut.

**Watson's lemma** says that if a series is an asymptotic expansion for $Z(\lambda)$, then for a fixed number of terms in the series the error is given by the first omitted term depending on the coupling constant $\lambda$. In that case, there exists an optimal choice of $n$ for a given $\lambda$.

One can improve the series' convergence by *Borel summation*. Supposing that

$$Z(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$$

is divergent, one defines

$$F(t) = \sum_{n=1}^{\infty} \frac{a_n}{n!} t^n \tag{17.16}$$

which is convergent in some cases, like the one above (there is a finite convergence radius). One can then rewrite $Z(\lambda)$ as

$$Z(\lambda) = \int_0^{\infty} e^{-t} F(t\lambda) dt \tag{17.17}$$

where $F(t\lambda)$ is to be continued analytically. **Exercise**: check eq. (17.17) by performing the $t$-integration over the summands.

### Example

The Euler-Heisenberg action in closed form is

$$S = -\frac{e^2 B^2}{8\pi^2} \int_0^{\infty} \frac{ds}{s^2} \left( \coth(s) - \frac{1}{s} - \frac{s}{3} \right) \exp\left[ -\frac{m^2 s}{eB} \right] \tag{17.18}$$

This describes the one-loop effective action induced by a charged scalar in a constant magnetic field. Expanding it in powers of $eB/m^2 = g$ gives

$$S = -\frac{e^2B^2}{2\pi^2}\sum_{n=0}^{\infty}\frac{B_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{2eB}{m^2}\right)^{2n+2}$$

(do not confuse the Bernoulli numbers $B_n$ with the magnetic field strenght $B$). So, the coefficients $c_n$ of the series $\sum_n c_n g^n$ are

$$\begin{aligned} c_n = & \frac{2^{2n}B_{2n+4}}{(2n+4)(2n+3)(2n+2)} = \\ & (-1)^{n+1}\frac{\Gamma(2n+2)}{8}\left[\frac{1}{\pi^{2n+4}}+\frac{1}{(2\pi)^{2n+3}}+\frac{1}{(3\pi)^{2n+2}}\right] \end{aligned} \tag{17.19}$$

- For a general series with coefficients of the form

  $$c_n \propto \beta^n\Gamma(\gamma n+\delta)$$

  the Borel sum approximation is

  $$f(g) \approx \frac{1}{g}\int_0^\infty \frac{ds}{s}\frac{1}{1+s}\left(\frac{s}{\beta\gamma}\right)^{\delta/\gamma}\exp\left[-\left(\frac{s}{\beta g}\right)^{1/2}\right]$$

  which is exactly eq. (17.18), using

  $$\sum_{n=1}^{\infty}\frac{-2s^3}{n^2\pi^2(s^2+n^2\pi^2)} = \coth(s) - \frac{1}{s} - \frac{s}{3}$$

  Simplifying further, one has

  $$f(g) = \sum_{n=0}^{\infty}(-1)^n n! g^n \approx \frac{1}{g}\int_0^\infty ds\frac{e^{-s/g}}{1+s}$$

Using the above, the first term of the Euler-Heisenberg action becomes

$$S_{\text{Borel}} = \frac{e^2B^2}{4\pi^6}\int_0^\infty ds\frac{s}{1+s^2/\pi^2}\exp\left[-\frac{m^2s}{eB}\right]$$

which is already a good approximation; in fact, it is much better than the series expansion.

For more on this topic, see Gerald Dunne's paper, to be found at **hep-th/0011036**.

# Chapter 18

# Symmetries and conservation laws

## 18.1 Inner symmetries

Remember the complex Klein-Gordon Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi$$

This one-field case can be generated to $n$ fields:

$$\mathcal{L} = \partial_\mu \Phi_i^* \partial^\mu \Phi_i - m^2 \Phi_i^* \Phi_i \qquad i = 1, \ldots, n \tag{18.1}$$

$\mathcal{L}$ is invariant under transformations

$$\begin{aligned} \Phi_i(x) &\to \Phi_i'(x) = U_{ij} \Phi_j(x) \\ \Phi_i^*(x) &\to \Phi_i^{*\prime}(x) = U_{ij}^* \Phi_j^*(x) \end{aligned}$$

with *unitary* matrices $U$: $U^\dagger U = U U^\dagger = \mathbb{1}$. Such a $U$ has $n^2$ generators $T^A$:

$$U = \exp(i\alpha^A T^A), \qquad A = 1, \ldots, n^2 \tag{18.2}$$

Here, the $T^A$ are chosen to be Hermitean (**exercise**: show that real fields would lead to orthogonal transformations $SO(n)$). Organizing the fields $\Phi_i$ into vectors $\Phi$, one sees the invariance of the Lagrangian density as follows:

$$\begin{aligned} \mathcal{L} = &\partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi \to \\ &\partial_\mu (U\Phi)^\dagger \partial^\mu (U\Phi) - m^2 (U\Phi)^\dagger (U\Phi) = \\ &\partial_\mu \Phi^\dagger (U^\dagger U) \partial^\mu \Phi - m^2 \Phi^\dagger (U^\dagger U) \Phi = \mathcal{L} \end{aligned}$$

Infinitesimal changes are given by

$$\begin{aligned} \delta \Phi_i &= (i\alpha^A T^A)_{ij} \Phi_j \\ \delta \Phi_i^* &= (-i\alpha^A T^A)_{ji} \Phi_j^* \end{aligned} \tag{18.3}$$

Now consider the variation of the Lagrangian density, which should be zero for such infinitesimal changes:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_i}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\delta(\partial_\mu\Phi_i) + \frac{\partial\mathcal{L}}{\partial\Phi_i^*}\delta\Phi_i^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i^*)}\delta(\partial_\mu\Phi_i^*) = 0$$

Rewriting the first term with the help of the Euler-Lagrange equation and using $\delta(\partial_\mu\Phi_i) = \partial_\mu(\delta\Phi_i)$, this yields

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i^*)}\delta\Phi_i^*\right) = 0$$

or

$$\boxed{\partial_\mu j^\mu = 0} \tag{18.4}$$

with the *Noether current*

$$j^\mu = \alpha^A(\partial^\mu\Phi_i^* T_{ij}^A\Phi_j - \partial^\mu\Phi_i T_{ij}^A\Phi_j^*) = \tag{18.5}$$

$$\alpha^A j^{A\mu} \tag{18.6}$$

for small $\alpha^A$, in the Klein-Gordon case. This $j^{A\mu}$ is also a conserved current, and the associated charge

$$Q^A = \int d^3x j^{A0} \tag{18.7}$$

is indeed conserved:

$$\boxed{\frac{d}{dt}Q^A = \int d^3x\frac{\partial}{\partial t}j^{A0} = \int d^3x\vec{\nabla}\cdot\vec{j}^A = 0} \tag{18.8}$$

where in the last step, we have applied Gauss' theorem. The existence of the conserved current and charge $j^{A\mu}$ and $Q^A$ is a consequence of **Noether's theorem**, which states that for every continuous symmetry, a conserved current and charge exist.

In the above example, we have studied a $U(N)$-symmetry, $U(N)$ standing for unitary matrices in $n$ dimensions. One can also take special unitary matrices, with $\det U = 1$; this corresponds to $SU(N)$-symmetries. Also note that here, we have taken space-independent matrices; we will make them space-dependent later on, which will lead us to gauge theories.

### Remark

The calculation of the conserved charge $Q^A$ is not unique in the sense that the integration may be carried out over any spacelike surface characterised by a 4-vector:

$$Q^{A\prime} = \int_S da_{3\mu} j^{A\mu} = Q^A$$

(This will not be proven here; it involves Gauss' theorem in four dimensions.)

## 18.2 Translational invariance, energy-momentum tensor

For scalar fields, we have the following translation rule

$$\begin{aligned} \Phi'(x') &= \Phi(x) \qquad \text{with} \qquad x' = x + a \qquad \text{so} \\ \Phi'(x) &= \Phi(x - a) \end{aligned}$$

For infinitesimal translations ($x' = x + \epsilon$),

$$\delta\Phi = -\epsilon^\mu \partial_\mu \Phi \tag{18.9}$$

The action $S = \int d^4x\mathcal{L}$ is invariant under such transformations. From the differential of the Lagrangian, we can again derive a conserved current:

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(x - \epsilon) - \mathcal{L}(x) \qquad \text{for a scalar field } \mathcal{L} \\ &\stackrel{!}{=} \frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta(\partial_\mu\Phi) \qquad \text{for real } \Phi \end{aligned}$$

Using eq. (18.9), we get

$$-\epsilon^\mu\partial_\mu\mathcal{L} = -\epsilon^\nu\partial_\mu g^\mu{}_\nu\mathcal{L} \stackrel{!}{=} \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}(-\epsilon^\nu\partial_\nu\Phi)\right)$$

so

$$\boxed{\epsilon^\nu\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - g^\mu{}_\nu\mathcal{L}\right) = 0} \tag{18.10}$$

The terms between the brackets define the canonical *energy-momentum tensor* $T^\mu_{\ \nu}$. It is the current $j^\mu$, where the extra index $\nu$ characterizes translations in direction $\nu$. As its name says, it contains the conserved 4-momentum of the field $\Phi$:

$$\begin{aligned} P^\nu &= \int d^3x T^{0\nu} = (H, \vec{P}), \qquad \text{in particular} \qquad (18.11) \\ P^0 &= H = \int d^3x(\Pi\partial^0\Phi - \mathcal{L}) \end{aligned}$$

In the Klein-Gordon, Dirac and Maxwell cases, it takes the following forms:

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\mathcal{L} = T_{\nu\mu} \qquad \text{(Klein-Gordon)} \\ T_{\mu\nu} &= \frac{i}{2}(\bar{\Psi}\gamma_\mu\partial_\nu\Psi - (\partial_\nu\bar{\Psi})\gamma_\mu\Psi \neq T_{\nu\mu} \qquad \text{(Dirac)} \qquad (18.12) \\ T_{\mu\nu} &= -F_{\mu\lambda}\partial_\nu A^\lambda + \frac{1}{4}g_{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda} - \frac{\lambda}{2}(2(\partial_{\bar{\mu}}A^{\bar{\mu}})\partial_\nu A_\mu - g_{\mu\nu}(\partial_{\bar{\mu}}A^{\bar{\mu}})^2) \\ &\qquad \text{(Maxwell)} \end{aligned}$$

The energy-momentum tensor retains its status as Noether current under addition of the total divergence of an antisymmetric tensor:

$$\Theta_{\mu\nu} = T_{\mu\nu} + \partial^\lambda (F_{[\mu\lambda]} A_\nu) \tag{18.13}$$

$\Theta_{\mu\nu}$ is symmetric in $\mu$ and $\nu$, and is called the *improved energy-momentum tensor*. Symmetrizing the energy-momentum tensor is relevant with regard to the Einstein equations in general relativity: the energy-momentum tensor appearing there has to be symmetric.

## 18.3 Lorentz invariance

A general Lorentz transformation is given by

$$x \to x' = \Lambda x$$

An infinitesimal Lorentz transformation has

$$x'^\mu = x^\mu + \frac{1}{2}\omega^{[\rho\sigma]}(x_\rho \delta^\mu_\sigma - x_\sigma \delta^\mu_\rho) \tag{18.14}$$

where $\omega^{[\rho\sigma]}$ is antisymmetric, leaving six free parameters. A field $\Phi_A$, in some representation of the Lorentz group with index $A$, transforms as follows:

$$\begin{aligned} \Phi'_A(x) &= S_{AB}(\Lambda)\Phi_B(\Lambda^{-1}x) \qquad \text{with} \\ S_{AB}(\Lambda) &= \left\{ \exp\left(-\frac{i}{2}\omega^{[\rho,\sigma]}\Sigma_{\rho\sigma}\right)\right\}_{AB} \end{aligned} \tag{18.15}$$

where the $\Sigma_{\rho\sigma}$ are a representation of the generators (the Lie algebra). The infinitesimal version is:

$$\begin{aligned} \Phi'_A(x) = \Phi_A(x) - \frac{i}{2}\omega^{[\rho,\sigma]}\Big( (\Sigma_{\rho\sigma})_{AB}\Phi_B(x) + (x_\rho\partial_\sigma - x_\sigma\partial_\rho)\Phi_A(x)\Big) = \\ \Phi_A(x) - \frac{i}{2}\omega^{[\rho,\sigma]}(\delta\Phi_A)_{\rho\sigma} \end{aligned} \tag{18.16}$$

All of this gives for $\delta\mathcal{L}$ (for a scalar $\mathcal{L}$):

$$\begin{aligned} \delta\mathcal{L} = &-\frac{1}{2}\omega^{[\rho,\sigma]}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\mathcal{L} = \\ &-\frac{1}{2}\omega^{[\rho,\sigma]}\partial_\mu\left\{(x_\rho\delta^\mu_\sigma - x_\sigma\delta^\mu_\rho)\mathcal{L}\right\} = \\ &-\frac{1}{2}\omega^{[\rho,\sigma]}\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}(\delta\Phi_A)_{\rho\sigma}\right\} \end{aligned}$$

Using the last identity (from the second to the third step) we get

$$\boxed{-\frac{1}{2}\omega^{[\rho,\sigma]}\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}(\delta\Phi_A)_{\rho\sigma}-(x_\rho\delta^\mu_\sigma-x_\sigma\delta^\mu_\rho)\mathcal{L}\right\}=0} \tag{18.17}$$

So, our conserved current is

$$M^\mu_{[\rho,\sigma]}=x_\rho T^\mu{}_\sigma-x_\sigma T^\mu{}_\rho+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}i(\Sigma_{\rho\sigma})_{AB}\Phi_B \tag{18.18}$$

The first two terms correspond to the orbital angular momentum density tensor, and the last one to the spin density tensor.

Applying the same trick as at the end of section 18.2, we can introduce the general *improved energy-momentum tensor*

$$\Theta^{\mu\nu}=T^{\mu\nu}+\frac{1}{2}\partial_\lambda F^{[\lambda\mu]\nu} \tag{18.19}$$

where

$$F^{[\lambda\mu]\nu}=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}(\Sigma^{\lambda\nu})_{AB}\Phi_B-\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\Phi_A)}(\Sigma^{\mu\nu})_{AB}\Phi_B \tag{18.20}$$

This allows us to rewrite the conserved current (eq. (18.19)) as

$$M^\mu_{[\rho\sigma]}=x_\rho\Theta^\mu{}_\sigma-x_\sigma\Theta^\mu{}_\rho$$

Considering only $M^\mu_{[i,j]}$, one finds conservation of angular momentum; similarly, $M^\mu_{[0,i]}$ gives conservation of center of mass velocity.

### Remarks

- After quantization (so far, we have worked with classical quantities), the transformation of field operators is achieved by unitary operators in Fock space:

  $$\mathbf{U}\Phi_A\mathbf{U}^\dagger=\Phi'_A$$

  The explicit construction of these **U**'s takes more effort, and will not be shown here.

- For a Dirac field, the orbital angular momentum tensor is given by

  $$L^\mu_{[\rho\sigma]}=x_\rho T^\mu_\sigma-x_\sigma T^\mu_\rho$$

  and the spin angular momentum tensor by

  $$S^\mu_{[\rho\sigma]}=\frac{i}{2}\bar{\Psi}(\gamma^\mu\Sigma_{\rho\sigma}+\Sigma_{\rho\sigma}\gamma^\mu)\Psi$$

with

$$\begin{aligned}\Sigma_{\rho\sigma} &= \frac{1}{4}\left[\gamma_\rho, \gamma_\sigma\right] = -\frac{i}{2}\sigma_{\rho\sigma} \qquad \text{and} \\ \Theta_{\mu\nu} &= \frac{i}{2}\bar{\Psi}\left(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu\right)\Psi\end{aligned}$$

The continuity equation can be checked with the help of the field equation. In the quantized case on has to take into account the fact that certain operators anticommute.

## 18.4 Discrete symmetries

Discrete symmetries are, for example, space or time inversions and charge conjugations.

### 18.4.1 Space inversion

Space inversion is achieved by the parity operation **P**:

$$\begin{aligned}\mathbf{P} : \vec{x} &\to -\vec{x} \\ \mathbf{P}^2 &= 1\end{aligned}$$

The transformation of a *scalar field* under space inversion is given by

$$\begin{aligned}\mathbf{P}\Phi(x)\mathbf{P}^\dagger &= \eta_P\Phi(x^0, -\vec{x}) \\ \mathbf{P}\Phi^\dagger(x)\mathbf{P}^\dagger &= \eta_P^*\Phi^\dagger(x^0, -\vec{x})\end{aligned} \tag{18.21}$$

with

$$|\eta_P| = 1 \qquad \text{and} \qquad \eta_P^2 = 1 \qquad \text{so} \qquad \eta_P = \pm 1$$

where **P** now is an operator in field space. This implies for the field components that

$$\begin{aligned}\mathbf{P}a(\vec{p})\mathbf{P}^\dagger &= \eta_P a(-\vec{p}) \\ \mathbf{P}b(\vec{p})\mathbf{P}^\dagger &= \eta_P b(-\vec{p})\end{aligned} \tag{18.22}$$

In case of parity invariance,

$$\mathbf{P}\mathcal{L}(\Phi(x))\mathbf{P}^\dagger = \mathcal{L}(\Phi(x^0, -\vec{x})) \tag{18.23}$$

For the Dirac *spinor field*, we have

$$\mathbf{P}\Psi(x^0, \vec{x})\mathbf{P}^\dagger = \gamma^0\Psi(x^0, -\vec{x}) \tag{18.24}$$

$\mathcal{L}_{\mathrm{D}}$ is parity invariant by construction (see the discussion of the representations of the Lorentz group). This $\mathbf{P}$ is explicity constructed in Björken & Drell II.

The electromagnetic field, finally, has

$$\mathbf{P}A^{\mu}(x)\mathbf{P}^{\dagger} = \Lambda^{P\mu}{}_{\nu}A^{\nu}(x^0, -\vec{x}) \tag{18.25}$$

A thorough discussion of discrete symmetries, including ray (or projective) representations, can be found in Weinberg's book, volume I.

### 18.4.2 Time reversal

Time reversal is accomplished by an operator $\mathbf{T}$ which sends $t$ to $-t$, or $x^0$ to $-x^0$. The field equation, in the Heisenberg picture,

$$-i\frac{\partial\Phi}{\partial t} = [H, \Phi]$$

should be *form*invariant under time reversal. With

$$\mathbf{T}\Phi(x^0, \vec{x})\mathbf{T}^{-1} = \eta_T\Phi(-x^0, \vec{x}), \qquad \eta_T = \pm 1 \tag{18.26}$$

and $\mathbf{T}$ unitary (i.e. $\mathbf{T}^{-1} = \mathbf{T}^{\dagger}$), we would need

$$\mathbf{T}\mathbf{H}\mathbf{T}^{\dagger} = -\mathbf{H}$$

which is impossible for $\mathbf{H}$, since this is a bilinear expression in $\Phi$. It would also imply negative energies for $\mathbf{H}$. So, $\mathbf{T}$ can not be unitary, but must be *anti-unitary* instead:

$$\mathbf{T} = \mathbf{J}\mathbf{U}$$

with $\mathbf{J}$ having the effect of complex conjugation (exchanging bras and kets) and $\mathbf{U}$ being unitary. This implies

$$\begin{aligned} \langle \mathbf{T}_{\mathrm{AU}}\phi | \mathbf{T}_{\mathrm{AU}}\Psi\rangle &= \langle \Psi|\phi\rangle \qquad \text{and} \\ \mathbf{T}_{\mathrm{AU}}(\alpha\phi) &= \alpha^*\mathbf{T}_{\mathrm{AU}}\phi \end{aligned}$$

This $\mathbf{T}_{\mathrm{AU}}$ changes the $i$ in the Heisenberg equation to $-i$, reducing our condition to $\mathbf{T}_{\mathrm{AU}}\mathbf{H}\mathbf{T}_{\mathrm{AU}}{}^{\dagger} = \mathbf{H}$.

On Dirac spinors, the time reversal operator has the effect

$$\mathbf{T}\Psi(x^0, \vec{x})\mathbf{T}^{-1} = \eta_T\mathbf{T}\Psi(-x^0, \vec{x})$$

It is defined by

$$\mathbf{T} = -i\gamma^1\gamma^3 = -i\gamma^5\mathbf{C} \tag{18.27}$$

(so $\mathbf{T} = \mathbf{T}^\dagger = \mathbf{T}^{-1} = -\mathbf{T}^*$). This form can be derived from the condition

$$\mathbf{T}\gamma_\mu \mathbf{T}^{-1} = \gamma_\mu^*$$

Note: in the unquantized Dirac equation, $\Psi \xrightarrow{\mathbf{T}} \mathbf{T}\Psi^*(-x^0, \vec{x})$; do not confuse the two.

For the *electromagnetic field*, time reversal takes the form of a proper antiorthochronous (i.e., space-preserving, time-reversing) Lorentz transformation:

$$\begin{aligned} \mathbf{T}A^\mu(x)\mathbf{T}^{-1} &= -\Lambda^\mu{}_\nu{}^{(T)} A^\nu(-x^0, \vec{x}) \\ \mathbf{T}j^\mu(x)\mathbf{T}^{-1} &= -\Lambda^\mu{}_\nu{}^{(T)} j^\nu(-x^0, \vec{x}) \end{aligned} \tag{18.28}$$

(For a direct construction of $\mathbf{T}$, see Björken & Drell, vol. 2.) **Exercise**: check the time-reversal invariance of the Maxwell-equations.

### 18.4.3 Charge conjugation

For charge conjugation, a similar operator exists, which acts on *scalar fields* like

$$\begin{aligned} \mathbf{C}\Phi\mathbf{C}^{-1} &= \eta_C \Phi^\dagger \\ \mathbf{C}Q\mathbf{C}^{-1} &= -Q \end{aligned} \tag{18.29}$$

It is a unitary matrix, and can be constructed (exercise).

On *Dirac fields*, its action is

$$\begin{aligned} \mathbf{C}\Psi(x)\mathbf{C}^{-1} &= \eta_C \mathbf{C}\bar{\Psi}^T(x) \\ \mathbf{C}a(k,s)\mathbf{C}^{-1} &= \eta_C b(k,s) \end{aligned} \tag{18.30}$$

As expected, it exchanges particles and antiparticles with the same momentum and sign. Its explicit form is $\mathbf{C} = i\gamma^0\gamma^2$, which in particular implies that $\mathbf{C}\bar{v}^{sT} = u^s$.

For *gauge fields*, to conclude, one has

$$\begin{aligned} \mathbf{C}A^\mu(x)\mathbf{C}^{-1} &= -A^\mu(x) \\ \mathbf{C}j^\mu\mathbf{C}^{-1} &= -j^\mu \end{aligned} \tag{18.31}$$

#### Note

In local QFT, with Lorentz invariance and the spin statistics relation, it can be shown that the combined transformation **CPT** leaves the Lagrangian (and hence the action) invariant (this is known as the *CPT-theorem*).

## 18.5 Ward identities, Goldstone bosons, spontaneous symmetry breaking

Above, we have demonstrated the Noether theorem: a continuous symmetry leads to a conserved *Noether current* and corresponding *Noether charge*. In the derivation, we used the classical equation of motion for the field. In the path integral approach, however, the fields that are to be summed do not fulfill this equation of motion. We will need a new technique, which will be developed below.

Consider a path integral for a complex scalar field:

$$I = \int \mathcal{D}\phi\mathcal{D}\phi^* \left\{\phi(x_1)\dots\phi(x_n)\phi^*(y_1)\dots\phi^*(y_m)\right\} e^{-S(\phi,\phi^*)}$$

(this is already in the Euclidean version). The global symmetry transformation

$$\phi \to e^{i\alpha}\phi, \qquad \phi^* \to e^{-i\alpha}\phi^* \tag{18.32}$$

with constant $\alpha$, which leaves $S$ invariant, can be extended to a local one, with an $x$-dependent $\alpha$. This can be interpreted as corresponding to a coordinate transformation which leaves $I$ invariant:

$$\begin{aligned}
0 = \delta I = &\int \mathcal{D}\phi\mathcal{D}\phi^* \left(\delta\left\{\phi(x_1)\dots\phi^*(y_m)\right\} e^{-S} + \left\{\phi(x_1)\dots\phi^*(y_m)\right\}\delta e^{-S}\right) = \\
&\int \mathcal{D}\phi\mathcal{D}\phi^* \left(\sum_{i=1}^{n}\int dxi\delta\alpha(x)\delta(x-x_i)\left\{\phi(x_1)\dots\phi^*(y_m)\right\} - \right.\\
&\qquad \sum_{j=1}^{m}\int dxi\delta\alpha(x)\delta(x-y_j)\left\{\phi(x_1)\dots\phi^*(y_m)\right\} + \\
&\qquad \left.\left\{\phi(x_1)\dots\phi^*(y_m)\right\}\int dx\delta\alpha(x)\partial_\mu j_\mu(x)\right) e^{-S}
\end{aligned}$$

with

$$\frac{\delta S}{\delta\alpha(x)} = -\partial_\mu\left(\frac{1}{i}\left\{\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right\}\right) = -\partial_\mu j_\mu \tag{18.33}$$

for

$$S = \int d^4x \left\{\partial_\mu\phi\partial_\mu\phi^* + m^2\phi^*\phi + V(\phi^*\phi)\right\} \tag{18.34}$$

(note that we do not use the equation of motion). For the above to be true, the coefficient of $\delta\alpha(x)$ must be zero; in other words,

$$\begin{aligned} i\sum_{i=1}^{n}\delta(x-x_i)\langle 0|\,\mathrm{T}(\phi(x_1)\ldots\phi^\dagger(y_m)\,|0\rangle - \\ i\sum_{j=1}^{m}\delta(x-y_j)\langle 0|\,\mathrm{T}(\phi(x_1)\ldots\phi^\dagger(y_m)\,|0\rangle = \\ -\partial_\mu\langle 0|\,\mathrm{T}(j^\mu(x)\phi(x_1)\ldots\phi^\dagger(y_m)\,|0\rangle \end{aligned} \tag{18.35}$$

This is a special version of the Ward-Takahashi identity, which is very useful in gauge theories, as we will see later.

Integrating this over a region $\Omega$ which contains all the $x_i$ and $y_j$ gives

$$\int_{\partial\Omega} dS^\mu \langle 0|\,\mathrm{T}(j_\mu(x)\phi(x_1)\ldots\phi^\dagger(y_m))\,|0\rangle = (n-m)\langle 0|\,\mathrm{T}(\phi(x_1)\ldots\phi^\dagger(y_m))\,|0\rangle$$

Extending $\Omega$ to infinity with naïve expectations about the behavior of the field $\phi(x)$ as $|x|\to\infty$, this would give zero, implying either $m$ to be equal to $n$ or the T-product vacuum expectation value to be zero. However, if $\phi(x)$ behaves differently in the infrared domain, there will be boundary terms. This is certainly the case if $\langle 0|\,\phi(x)\,|0\rangle \neq 0$ (this means $n=1$, $m=0$) i.e. if the symmetry (18.32) is *broken spontaneously*, i.e., the ground state no longer has the symmetries of the Lagrangian. In this situation, *Goldstone bosons* will appear; these are massless bosons which show up in the infrared.

The existence of Goldstone bososns can be proven rigorously, preferably by going back to the canonical formalism. Be careful, however:

$$Q = \int d^3x\, j^0$$

does not exist in this case (if there are no long-range forces, e.g. from gauge bosons). We will consider this later (see Higgs mechanism, chapter 23); here, we will first present a simple argument.

The $U(1)$-symmetry ($\phi \to e^{i\alpha}\phi$) is broken spontaneously by a constant vacuum expectation value $\phi_0 = \langle\phi\rangle$. Since $\phi_0$ does not enter the kinetic term, consider the potential; take the saddle point approximation for the path integral for $\phi$ correlators:

$$\left.\frac{\partial V}{\partial\phi}\right|_{\phi_0} = 0$$

for example for the so-called "Mexican hat" potential:

$$V(\phi^*\phi) = \frac{\lambda}{2}(\phi^*\phi-\mu^2)^2$$

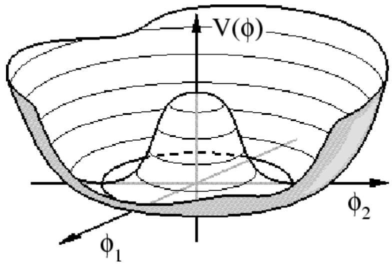



Figure 18.1: Mexican hat potential

Without loss of generality, one can choose $\phi_0$ to be real. Expanding $V$ around $\Phi_0$ as $\Phi = \Phi_R + i\Phi_I$, one gets

$$V(\phi^*\phi) = V(\phi_0^2) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial \phi_R^2}\right|_{\phi_0}(\delta\phi_R)^2 + \left.\frac{\partial^2 V}{\partial\phi_R\partial\phi_I}\right|_{\phi_0}\delta\phi_R\delta\phi_I + \frac{1}{2}\left.\frac{\partial^2 V}{\partial \phi_I^2}\right|_{\phi_0}(\delta\phi_I)^2$$

Writing $\delta\phi$ as an infinitesimal symmetry transformation ($\phi \to \phi(1+i\alpha)$), one has

$$\begin{aligned} \phi_R = \phi_0, &\quad \delta\phi_R = 0, \\ \phi_I = 0, &\quad \delta\phi_I = \phi_0\alpha, \end{aligned}$$

so

$$\frac{1}{2}\left.\frac{\partial^2 V}{\partial \phi_I^2}\right|_{\phi_0}\phi_0^2\alpha^2 = 0$$

i.e., $m_I^2 = 0$ for a Goldstone boson field $\Phi_I$ in the valley of a Mexican hat potential.

**Goldstone's theorem** states:

| |
|---|
| If the ground state of the theory does not have the symmetry of the Lagrangian, i.e. if the Lagrangian's symmetry is broken spontaneously, and if the interactions in the theory become short-ranged, but Poincaré invariance is not broken, there exists a massless boson, called (Nambu-)Goldstone boson, with the quantum number of the broken symmetry generator. |

The general proof will be only indicated here. A noninvariant vacuum means that there exists an operator field $A$ with

$$\delta a(t) = \lim_{V\to\infty}\langle 0|\,[Q_V(t), A(y)]\,|0\rangle = \langle 0|\underbrace{B(y)}_{\delta A}|0\rangle \neq 0 \quad \text{with}$$

$$Q_V = \int_V d^3x j^0(x^0, \vec{x})$$

with a conserved Noether current $\partial_\mu j^\mu = 0$ corresponding to some symmetry of the Lagrangian. (Here, we work with a fixed $V \neq \mathbb{R}^3$ to avoid divergences:

$$Q\,|0\rangle \neq 0 \quad \text{so}$$
$$\langle 0|\, Q^2\,|0\rangle = \int d^3x\, \langle 0|\, j^0(x) Q\,|0\rangle \overset{\text{transl. inv.}}{=} \int d^3x\, \langle 0|\, j^0(0) Q\,|0\rangle = \infty$$

diverges.)

In the example above, $A$ corresponds to $\phi$ and $B$ to $\delta\phi$. Let us evaluate the Ward-Takahashi identity (eq. (18.35)) with just one $\phi$:

$$i\delta(x - x_1)\underbrace{\langle 0|\, \phi(x)\,|0\rangle}_{\neq 0} = -\partial_\mu \langle 0|\, \mathrm{T}(j^\mu(x)\phi(x_1))\,|0\rangle \tag{18.36}$$

The spectral representation (due to Källen and Lehmann) gives in this case

$$\partial_\mu \langle 0|\, \mathrm{T}(j^\mu(x)\phi(x_1))\,|0\rangle = \partial_\mu \int_0^\infty d\sigma^2 \rho(\sigma^2)\partial^\mu \Delta_F(x - x_1, \sigma^2) \tag{18.37}$$

(we can set $x_1 = 0$; $\sigma^2$ is the mass variable). Fourier transforming eq. (18.36) and taking the limit $p \to 0$ gives

$$\langle 0|\, \delta\phi\,|0\rangle = i\phi_0 = \lim_{p\to 0} \int d\sigma^2 \frac{\rho(\sigma^2)(-p^2)}{\sigma^2 - p^2 - i\epsilon}$$

i.e., we need a $\delta(\sigma^2)$ contribution, a massless intermediate state. This must be a boson because of the Lorentz covariance of the left-hand side of eq. (18.37):

$$\langle 0|\, j^\mu(0)\,|p(m=0)\rangle \neq 0 = if\pi \frac{p^\mu}{p^2}$$
$$\langle p(m=0)|\, \phi(0)\,|0\rangle \neq 0$$

Here, $\langle 0|\, \delta\phi\,|0\rangle$ is called *order parameter*. The infinitesimal parameter $\delta\alpha$ is taken out of the brackets, as in, e.g., magnetism.

More extensive literature on this topic can be found in: Kugo, Itzykson & Zuber, Pokorski.

# Chapter 19

# Gauge theories: QED, QCD, QFD

## 19.1 Classical gauge covariant field equations

Let us start our discussion of gauge theories with classical gauge covariant field equations. Consider, for example, the Dirac Lagrangian

$$\mathcal{L}_{\mathrm{D}} = \bar{\Psi}(i\gamma_\mu\partial^\mu - m)\Psi \tag{19.1}$$

with $\Psi = \Psi_\alpha(x)$ a Dirac spinor. This Lagrangian has the *global* symmetry

$$\Psi_\alpha(x) \to e^{i\varphi}\Psi_\alpha(x) \tag{19.2}$$

for which the Noether theorem predicts a conserved current and associated charge. However, a global symmetry is undesirable from a relativistic point of view: it means that the same gauge transformation has to be chosen at every point in space. This comes down to something like a coordinate system extending over all of space, which is typically impossible in general relativity. A *local* symmetry would suit us much better:

$$\boxed{\Psi_\alpha(x) \to e^{i\varphi(x)}\Psi_\alpha(x)} \tag{19.3}$$

This is a $U(1)$-symmetry. To make our Lagrangian invariant under this symmetry, we have to introduce a gauge field $A_\mu(x)$ by replacing the normal derivative by a covariant one:

$$\boxed{\partial_\mu \to \partial_\mu - iA_\mu(x) = \mathcal{D}_\mu} \tag{19.4}$$

This procedure is called *minimal coupling.* With transformation (19.3) and the requirement that $A_\mu(x)$ transforms with (19.3) as

$$\boxed{A_\mu(x) \to A_\mu(x) + \partial_\mu\varphi(x)} \tag{19.5}$$

our Lagrangian is invariant:

$$\begin{aligned}\mathcal{L}_{\mathrm{D}} \to \mathcal{L}'_{\mathrm{D}} = &\bar{\Psi}'(i\gamma_\mu(\partial^\mu - iA'_\mu(x)) - m)\Psi' = \\ &\bar{\Psi}e^{-i\varphi(x)}\{i\gamma_\mu[\partial^\mu - i(A^\mu(x) + \partial^\mu\varphi(x))] - m\}e^{i\varphi(x)}\Psi = \\ &\bar{\Psi}e^{-i\varphi(x)}\{i\gamma_\mu[(\partial^\mu i\varphi(x)) - i((A^\mu(x) + (\partial^\mu\varphi(x)))] - m\}e^{i\varphi(x)}\Psi + \\ &\bar{\Psi}e^{-i\varphi(x)}\{i\gamma_\mu e^{i\varphi(x)}\partial^\mu\}\Psi = \mathcal{L}_{\mathrm{D}}\end{aligned}$$

Specifically for Maxwell theory, where $A_\mu(x)$ and $\varphi(x)$ are small, one takes out the small factor $e$, the electromagnetic coupling constant:

$$\begin{aligned}A_\mu(x) &= eA_\mu^{\mathrm{usual}}(x) \\ \varphi(x) &= e\varphi^{\mathrm{usual}}(x)\end{aligned}$$

The kinetic term for the gauge field is obtained by commuting the covariant derivative with itself:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -i(\partial_\mu A_\nu - \partial_\nu A_\mu) = -iF_{\mu\nu} \tag{19.6}$$

The Lagrangian of electrodynamics is given by

$$\mathcal{L}_{\mathrm{ED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{19.7}$$

where $A_\mu^{\mathrm{usual}}$ has been used. If one uses $A_\mu$ with the factor of $e$, the Lagrangian becomes $\mathcal{L}_{\mathrm{ED}} = -F^2/4e^2$.

## 19.2 Non-abelian gauge theories, Yang-Mills theory

Above, we considered a gauge transformation from an abelian group. Now, let us extend this discussion to non-abelian groups. The Lagrangian

$$\mathcal{L} = \bar{\Psi}_A(i\gamma\partial - m)\Psi_A \qquad \text{with } A = 1, \ldots, n \tag{19.8}$$

has a $U(n) = SU(n) \times U(1)$ symmetry, i.e. it is invariant under transformations of the type

$$\boxed{\Psi_A \to U_A{}^B\Psi_B} \tag{19.9}$$

For $SU(2)$, $U$ is

$$U = e^{i\vec{\tau}\cdot\vec{\varphi}/2} \tag{19.10}$$

with $\vec{\tau}$ the generators of the group, in this case the Pauli matrices

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since this is a $SU(2)$-transformation, the determinant is unity. In this case, this can also be seen from the identity $\det e^A = e^{\mathrm{tr}A}$ and the fact that the Pauli matrices are traceless.

For $SU(3)$,

$$U = e^{i\lambda_a \varphi^a/2}, \qquad a = 1, \ldots, 8 \tag{19.11}$$

where the $\lambda_a$ are the *Gell-Mann matrices*:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Note that with the exception of $\lambda_8$, all of these can be derived from the Pauli matrices by adding a column and a row of zeroes. Also note that they are again traceless, and Hermitean, and that only $\lambda_3$ and $\lambda_8$ have diagonal entries. Furthermore, the following relations between them hold:

$$\mathrm{tr}(\lambda_a \lambda_b) = 2\delta_{ab}, \quad \sum_a \lambda_a \lambda_a = \frac{16}{3}\mathbb{1}, \quad \sum_a \lambda_a \lambda_b \lambda_a = -\frac{2}{3}\lambda_b \quad \text{and}$$
$$\boxed{[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c} \tag{19.12}$$

The $\lambda_a$ are generators of $SU(3)$, and they are often normalized to get rid of the factor of 2 in eq. (19.12):

$$T_a = \frac{\lambda_a}{2} \tag{19.13}$$

The $T_a$ form the Lie algebra $\mathfrak{su}(3)$. The $f_{abc}$ are the *structure constants* of this Lie algebra, and can be written as an $8 \times 8$-matrix $(F^a)_{bc}$ (for $SU(2)$, the structure constant matrix is the three-dimensional Levi-Cività symbol $\epsilon_{ijk}$).

Other Lie algebras that play a role in gauge theories are those corresponding to the groups $SU(n), SO(n), E_6, E_7, E_8$ and others. For more on this topic, see the book by Fuchs and Schweigert (CUP).

### 19.2.1 Quantum chromodynamics

The gauge group of QCD is $SU(3)$. Quarks come in triplets, three-vectors on which the $SU(3)$-transformations act:

$$q_j = \begin{pmatrix} q_1^j \\ q_2^j \\ q_3^j \end{pmatrix}$$

The lower index, here written as taking the values 1, 2, or 3, represents the color; its values are often called red, green and blue. $j$ is the flavor index.

### 19.2.2 Electroweak theory

Electroweak theory, also known as quantum flavordynamics, works with doublets like

$$\begin{pmatrix} u \\ d' \end{pmatrix}_l$$

consisting of quark pairs. These doublets are left-handed, and can be projected out of a state by $(\mathbb{1} + \gamma_5)$. The symmetry group is $SU(2)$.

### 19.2.3 Lagrangian

For both theories, like with QED, the (global) symmetry is promoted to a local symmetry, also known as gauge symmetry:

$$\Psi(x) \to U(x)\Psi(x) \qquad (\Psi'_{A\alpha}(x) = U_A^B \Psi_{B\alpha}(x))$$

Again, a gauge field $\mathbb{A}_\mu$ is needed:

$$\partial_\mu \mathbb{1} \to \partial_\mu \mathbb{1} - i\mathbb{A}_\mu = \mathcal{D}_\mu \tag{19.14}$$

And again the gauge field needs a gauge transformation:

$$\begin{aligned} \mathbb{A}'_\mu(x) &= U\mathbb{A}_\mu(x)U^\dagger + iU\partial_\mu U^\dagger \\ &= iU(\mathcal{D}_\mu U^\dagger) \end{aligned} \tag{19.15}$$

The gauge transformation can also be written as a transformation for the covariant derivative:

$$\begin{aligned} \mathcal{D}'_\mu &= \partial_\mu - i\mathbb{A}'_\mu = \partial_\mu - i(iU(\mathcal{D}_\mu U^\dagger)) = \\ &= U\mathcal{D}_\mu U^\dagger \end{aligned} \tag{19.16}$$

With these transformation rules, the Lagrangian is again invariant:

$$\begin{aligned}
&\bar{\Psi}\{i\gamma^\mu(\partial_\mu - i\mathbb{A}_\mu) - m\}\Psi \to \\
&\qquad \bar{\Psi}'\{i\gamma^\mu(\partial_\mu - i\mathbb{A}'_\mu) - m\}\Psi' = \\
&\qquad \bar{\Psi}U^\dagger\{i\gamma^\mu[\partial_\mu - i(U\mathbb{A}_\mu U^\dagger + iU\partial_\mu U^\dagger)] - m\}U\Psi = \\
&\qquad \bar{\Psi}U^\dagger\{i\gamma^\mu[U\partial_\mu + (\partial_\mu U) - iU\mathbb{A}_\mu U^\dagger U - (\partial_\mu U)U^\dagger U] - mU\}\Psi = \\
&\qquad \bar{\Psi}\{i\gamma^\mu(\partial_\mu - i\mathbb{A}_\mu) - m\}\Psi
\end{aligned}$$

In analogy to the QED case, the kinetic term for the gauge field is obtained by commuting the covariant derivative with itself:

$$i[\mathcal{D}_\mu, \mathcal{D}_\nu] = \mathbb{G}_{\mu\nu} \tag{19.17}$$

$$\boxed{\mathbb{G}_{\mu\nu} = \partial_\mu\mathbb{A}_\nu - \partial_\nu\mathbb{A}_\mu - i[\mathbb{A}_\mu, \mathbb{A}_\nu]} \tag{19.18}$$

With eq. (19.16), we see that $\mathbb{G}_{\mu\nu}$ transforms homogeneously like a tensor under a gauge transformation:

$$\mathbb{G}'_{\mu\nu} = U\mathbb{G}_{\mu\nu}U^\dagger \tag{19.19}$$

This means that it is *covariant*, not *invariant*, under such transformations; classical electrodynamics is *invariant*. The trace of $\mathbb{G}_{\mu\nu}\mathbb{G}^{\mu\nu}$ is both Lorentz and gauge *invariant*.

Again, a coupling constant, this time called $g$, can be taken out of the gauge field:

$$\begin{aligned}
\mathbb{A}_\mu &\to g\mathbb{A}^{\text{us}}_\mu \qquad \text{and} \\
\mathbb{G}^{\text{us}}_{\mu\nu} &= \partial_\mu\mathbb{A}^{\text{us}}_\nu - \partial_\nu\mathbb{A}^{\text{us}}_\mu - ig[\mathbb{A}^{\text{us}}_\mu, \mathbb{A}^{\text{us}}_\nu]
\end{aligned}$$

This gives the gauge field's Lagrangian as

$$\mathcal{L}_\text{g} = \alpha\,\text{tr}(\mathbb{G}_{\mu\nu}\mathbb{G}^{\mu\nu}) = \alpha g^2\,\text{tr}(\mathbb{G}^{\text{us}}_{\mu\nu}\mathbb{G}^{\mu\nu,\text{us}})$$

where we choose $\alpha = -1/2g^2$. Writing the gauge field in the basis of the generators introduces the field strength components:

$$\begin{aligned}
\mathbb{A}_\mu &= A^a_\mu T_a \\
A^a_\mu &= 2\text{tr}(\mathbb{A}_\mu T^a)
\end{aligned} \tag{19.20}$$

(for the last identity, use $\text{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$). In terms of these field strengths, the Lagrangian becomes

$$\boxed{\mathcal{L}_\text{g} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a}} \tag{19.21}$$

(from now on, the superscript 'us', for 'usual', will be suppressed, and $\mathbb{A}$ will stand for $\mathbb{A}^{\mathrm{us}}$). Unlike in the classical electrodynamics case, the coupling constant $g$ is contained in the kinetic term now; this means that it is universal. In classical electrodynamics, one could choose different $e$'s for different sectors *(?)*. Embedding $U(1)$ in $U(n)$ avoids this arbitrariness, with the well-known result that the coupling constant in QED is also universal. Charges always have two functions: they are a conserved quantity, and they mediate the coupling between the field and the particle.

**Exercise**: write eq. (19.20) in terms of $A_\mu^a$.

### 19.2.4 Infinitesimal gauge transformations

Studying infinitesimal gauge transformations is convenient, since the non-linearity, which makes finite transformations complicated, is simplified. An infinitesimal transformation is given by

$$U \approx \mathbb{1} + \frac{i\lambda_a}{2}\varphi_a(x) \tag{19.22}$$

where $\varphi_a(x)$ itself is also infinitesimal. Under this transformation, $\mathbb{A}_\mu$ transforms into

$$\begin{aligned} \mathbb{A}_\mu^{\mathrm{us}} \to \frac{i}{g}U\mathcal{D}_\mu U^\dagger &= \frac{i}{g}(-ig\mathbb{A}_\mu) - \frac{1}{g}\left[\frac{\lambda_a\varphi_a}{2}, \mathcal{D}_\mu\right] = \\ &= \mathbb{A}_\mu + i\left[\frac{\lambda_a}{2}, A_\mu^b\frac{\lambda^b}{2}\right]\varphi_a(x) + \frac{1}{g}\partial_\mu\varphi_a(x)\lambda_a \end{aligned} \tag{19.23}$$

Taking the trace of $\mathbb{A}_\mu T^a$ leads to the following transformation rule for the field strength constants:

$$\boxed{A_\mu^a \to A_\mu^a - f^{bca}\varphi_b A_\mu^c + \frac{1}{g}\partial_\mu\varphi_a(x)} \tag{19.24}$$

#### Complete QCD Lagrangian

In the absence of a $\Psi$-field, one has (pure) Yang-Mills theory. If one does include this, one obtains the complete Lagrangian for quantum chromodynamics:

$$\boxed{\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2}\mathrm{tr}(\mathbb{G}_{\mu\nu}\mathbb{G}^{\mu\nu}) + \sum_{j=1}^{f}\bar{\Psi}^{(j)}(i\gamma^\mu\mathcal{D}_\mu - m_j)\Psi^{(j)}} \tag{19.25}$$

Varying this with respect to $\mathbb{A}_\mu$ and $\bar{\Psi}$, respectively, yields the field equations:

$$(i\gamma^\mu \mathcal{D}_\mu - m_j)\Psi^{(j)}(x) = 0 \tag{19.26}$$

$$[\mathcal{D}^\mu, \mathbb{G}_{\mu\nu}] = \sum_a (\bar{\Psi}\gamma_\nu \lambda^a \Psi)\frac{\lambda_a}{2} \tag{19.27}$$

These are non-linear equations in $\mathbb{A}_\mu$, just like in general relativity the equations for the Christoffel symbols (the analogue of the $A^a_\mu$) are non-linear.

## 19.3 Path integral for gauge theories, gauge fixing

Naïvely, one could assume the generating functional for a gauge theory (without fermions in this case) to be given by

$$Z(j) = N \int \mathcal{D}\mathbb{A}_\mu \exp\left\{ i \int d^4x (\mathcal{L} + \mathrm{tr}[j^\mu \mathbb{A}_\mu]) \right\} \tag{19.28}$$

where $j^\mu$ is a color matrix. However, although $\mathcal{L}$ is gauge invariant, the same physical field is represented by a whole class of vector fields $\mathbb{A}_\mu$, which are related to each other by gauge transformations. In other words, with this definition of the generating functional, the same field is summed over several times (infinitely many, to be 'precise'). For example, the electrodynamic $A^T_\mu = A_\mu - (\partial_\mu/\partial^2)(\partial^{\mu'} A_{\mu'})$ is gauge invariant under the transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$.

To make matters worse, the weighting by integration over orbits might be different for different physical $A^T$. In conclusion, the separation of a simple volume factor for the orbits is generally not sufficient. To address this problem,

(i) we have to study *gauge fixing*

(ii) we have to consider *zero modes*: in the exponential, certain parts of $A_\mu$ do not enter the bilinear part of $\mathcal{L}$; these parts have to be integrated separately, or rather, eliminated

There is a relation between (i) and (ii):

$\boxed{\textit{Gauge fixing is equivalent to the elimination of the zero modes}}$

This equivalence can only be seen immediately in simple cases, such as QED:

$$\begin{aligned}\int d^4x\mathcal{L}_{\text{QED}}^{\text{free}} = & -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \\ & -\frac{1}{4}\int d^4x(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \\ & \frac{1}{2}\int d^4x(A_\nu \partial^2 A^\nu - A_\nu \partial_\mu \partial^\nu A^\mu) = \\ & \frac{1}{2}\int d^4x A_\nu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\mu\end{aligned}$$

Applying a Fourier transform turns the factor between brackets into a matrix $\tilde{M}^{\mu\nu}(k)$:

$$\tilde{M}^{\mu\nu}(k) = -(g^{\mu\nu}k^2 - k^\mu k^\nu)$$

Since $\tilde{M}^{\mu\nu}(k)k_\mu = 0$, the generating functional has a zero mode $k_\mu\tilde{\Lambda}$, which is generated by a gauge transformation

$$\tilde{A}_\mu \to \tilde{A}_\mu + k_\mu\tilde{\Lambda}$$

**Remarks**

- $\tilde{M}^{\mu\nu}$ is the Fourier transform of a *quasilocal* operator $M_{\mu\nu}(x,y)$:

$$\int d^4x A_\mu(x) M_x^{\mu\nu} A_\nu(x) = \int d^4x d^4y A_\mu(x) \underbrace{M_x^{\mu\nu}\delta^4(x-y)}_{M^{\mu\nu}(x,y)} A_\nu(y)$$

- We have to eliminate as many gauge degrees of freedom as there are zero modes in order to allow for an inverse of $M$; this leads to *complete* gauge fixing, which is a complicated business involving Gribov copies (for a more detailed treatment of this topic, please consider the literature).

- The bilinear part alone is *not* gauge invariant.

### 19.3.1 Gauge fixing scheme

One begins with a symmetric $N \times N$-matrix $A$ with $n$ zero modes. Let $x_i \to y_i$ be an orthogonal transformation to a coordinate system where $A$ is diagonal:

$$x_i A_{ik} x_k = y_i A_{ii}^{\text{diag}} y_i \tag{19.29}$$

Now define the following integral, which will turn out to be a primitive version of the path integral:

$$G(A) = \int dy_1 \dots dy_{N-n} \exp(-x^T(y)\, A\, x(y)) \tag{19.30}$$

Note that $y_{N-n+1} \dots y_N$ do not appear in the exponent; these are the zero modes. Now choose some $y_{N-n+1} \dots y_N$ and rewrite the integral:

$$G(A) = \int dy_1 \dots y_N \delta(y_{N-n+1}) \dots \delta(y_N) \exp(-x^T(y)Ax(y))$$

The $\delta$-functions are called *gauge conditions*. With this reformulation, the integral measure can be rewritten as

$$dy_1 \dots dy_N = dx_1 \dots dx_N \det \left( \frac{\partial y}{\partial x} \right)$$

which does not depend on the choice of $y_{N-n+1} \dots y_N$, as long as the determinant is nonzero. Note also that the orthogonal transformation does not have to be performed explicitly.

#### Note

From the eigenvalue equation for $\tilde{M}^{\mu\nu}(k)$

$$g_{\lambda\mu} \tilde{M}^{\mu\nu}(k) \tilde{V}_\nu(k) = m \tilde{V}_\lambda(k)$$

one can see that $(\tilde{M}^{-1})^{\mu\nu}$ does not exist, since $m = 0$ for zero modes, so

$$\det \tilde{M}^{\mu\nu}(k) = 0 \quad \forall k$$

### 19.3.2 Examples

#### Lorenz gauge

Let us use the Lorenz (note: not Loren*tz*) gauge in order to remove the zero modes:

$$\partial^\mu A_\mu = 0$$

(in QED). Now, however, we still can have a $\Lambda$ with $\partial^2 \Lambda = 0$ (so $k^2 \tilde{\Lambda} = 0$), so the gauge is not completely fixed yet. The Fourier transform of this $\Lambda$ is then given by

$$\tilde{\Lambda} = a\delta(k^2) + a^\mu \frac{\partial}{\partial k^\mu} \delta(k^2) \tag{19.31}$$

(The $\delta$-function is there since only the $k^2 = 0$ mode contributes.) In the Euclidean case, $k^2 = 0$ implies $k^\mu = 0$, and because of Lorentz invariance, we can boost to a frame where $k^\mu$ goes to $(k_0 \ \vec{0})^T$. To complete the gauge fixing, we need an additional term in $\mathcal{L}$: a constraint with a Lagrange multiplier, namely $\frac{1}{\alpha}(\partial_\mu A^\mu)^2$. This leads to a change in $\tilde{M}^{\mu\nu}(k)$:

$$\tilde{M}^{\mu\nu}(k) = -k^2 g^{\mu\nu} + k^\mu k^\nu - \frac{1}{\alpha} k^\mu k^\nu$$

Now, after the above-mentioned boost, multiplication with $k_\nu$ does not give zero if $k^2 \neq 0$.

$$\det \tilde{M} = \det \left( -k_0^2 g^{\mu\nu} + \left( \begin{array}{cc} (1-\frac{1}{\alpha})k_0^2 & 0 \\ 0 & 0 \end{array} \right) \right) \neq 0$$

if $k_0^2 \neq 0$ and $\alpha \neq \infty$.

### Coulomb gauge

In QED, the Coulomb gauge,

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad (\vec{A} \to 0 \text{ for } |\vec{x}| \to \infty) \tag{19.32}$$

completely fixes $\Lambda$. In QCD, Gribov ambiguities arise if $gA$ is large.

### Axial gauge

The axial gauge is defined by

$$n^\mu A_\mu = 0$$

(For $n^2 = 0$, this is the 'light cone gauge', and for $\vec{n} = 0$ the 'temporary gauge'.) In QED, this means that $n^\mu \partial_\mu \Lambda = 0$, so $\tilde{\Lambda} \propto \delta(k \cdot n)$. The QCD case is more complicated.

## 19.4 Faddeev-Popov (B. deWitt, Feynman) procedure

Simply writing the gauge fixing conditions (see examples above)

$$\boxed{\mathcal{F}_b(A^a_\mu) = 0} \tag{19.33}$$

as $\delta$-functions into the path integral is too naïve: we need a proper weighting of the orbits in order not to destroy the gauge invariance of the path integral. The Faddeev-Popov procedure provides us with such a weighting; it works as follows.

- $\int \mathcal{D}A^a_\mu$ is *gauge invariant*, but does sum the same physical field several times.
- The gauge transformation

$$\begin{aligned} \mathbb{A}^g_\mu &= U_g^{-1} \mathbb{A}_\mu U_g + \frac{i}{g} U_g^{-1} \partial_\mu U_g && (19.34) \\ d\mathbb{A}^g_\mu &= U_g^{-1} d\mathbb{A}_\mu U_g && (19.35) \end{aligned}$$

  is a unitary transformation, so its Jacobian is unity.

- The integral

$$\int \underbrace{\mathcal{D}g}_{\prod_x dg(x)} \prod_{x,b} \delta\left(\mathcal{F}_b(A_\mu^{a\,g}(x))\right) \equiv \Delta^{-1}(A_\mu^a) \tag{19.36}$$

is also gauge invariant (this is rather trivial: if we integrate over all symmetry transformations, the result must be symmetric). Proof:

$$\Delta^{-1}((A_\mu^a)^{g_0}) = \int \mathcal{D}g \prod_{x,b} \delta(\mathcal{F}_b((A_\mu^{a\,g_0}(x))^g)) \stackrel{!}{=}$$

$$\int \mathcal{D}(g\,g_0) \prod_{x,b} \delta(\mathcal{F}_b((A^{g\,g_0})_\mu^a(x))) = \Delta^{-1}(A_\mu^a)$$

because of the invariance of the *Haar measure* of integration.

**Haar measure**

- The Haar measure is "right invariant":

$$dg = d(g\,g_0) \tag{19.37}$$

E.g. for $SU(2)$,

$$g(x) = e^{i\vec{\alpha}\cdot\vec{\sigma}/2} = \cos\left(\frac{\alpha}{2}\right)\mathbb{1} + i\vec{\sigma}\cdot\hat{\vec{\alpha}}\sin\left(\frac{\alpha}{2}\right)$$

The parameter space is a sphere with radius $2\pi$ (redefining $\alpha/2 \to \alpha$), or the surface of a 4-dimensional sphere:

$$\begin{pmatrix} \sin\alpha\hat{\vec{\alpha}} \\ \cos\alpha \end{pmatrix} \qquad \text{with} \qquad \hat{\vec{\alpha}} = \begin{pmatrix} \sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix}$$

with the invariant surface element

$$d\Omega_4 = \sin\beta\, d\beta\, d\gamma \sin^2\alpha\, d\alpha$$

For $U(1)$, it is trivial:

$$dg = d\varphi = d(\varphi + \varphi_0) = d(g\,g_0)$$

Continuing with our gauge fixing procedure, we insert $1 = \Delta\Delta^{-1}$ into our naïve expression for $Z$:

$$Z = \int \mathcal{D}A_\mu^a\, \Delta(A_\mu^a) \int \mathcal{D}g \prod_{x,b} \delta(\mathcal{F}_b(A_\mu^a(x)^g))e^{iS(A)}$$

(Note that the sources $j^a_\mu$ coupling to $A^a_\mu$ are not gauge invariant.) Interchanging the integrations gives

$$Z = \int \mathcal{D}g \int \mathcal{D}A^a_\mu \, \Delta(A^a_\mu) \prod_{x,b} \delta(\mathcal{F}_b(A^a_\mu(x)^g))e^{iS(A)}$$

$\int \mathcal{D}g$, $\Delta(A^a_\mu)$ and $e^{iS(A)}$ are all gauge invariant, so the only remaining factor, $\prod_{x,b} \delta((A^g)^a_\mu(x))$, must be equal to $\prod_{x,b} \delta(A^a_\mu(x))$. This gives us for $Z$:

$$Z = \int \mathcal{D}g \int \mathcal{D}A^a_\mu \Delta(A^a_\mu) \prod_{x,b} \delta(\mathcal{F}_b(A^a_\mu(x)))e^{iS(A)} \tag{19.38}$$

where the integration over $\mathcal{D}g$ is like an overall "volume" factor.

Now, we can calculate $\Delta(A^a_\mu)$ with the assumption that $A^a_\mu$ fulfills the gauge conditions. Remember the definition of $\Delta^{-1}(A^a_\mu)$:

$$\Delta^{-1}(A^a_\mu) := \int \mathcal{D}g \prod_{x,b} \delta(\mathcal{F}_b(A^a_\mu(x)^g))$$

Since we assume that $A^a_\mu$ fulfills $\mathcal{F}_b(A^a_\mu) = 0$, we see that only $g = 1$ contributes. Going to an infinitesimal gauge transformation, we have

$$\begin{aligned} U_g = \mathbb{1} + i\vec{T} \cdot \vec{\epsilon} \qquad &\text{with} \qquad \vec{T} = \vec{\lambda}/2 \\ \mathbb{A}^g_\mu &\approx \mathbb{A}_\mu - \frac{i^2}{g}[\mathcal{D}_\mu, \vec{T} \cdot \vec{\epsilon}] \end{aligned}$$

(Remember, $\mathbb{A}^g_\mu = \frac{i}{g} U_g \mathcal{D}_\mu U^\dagger_g$.)

Writing out the transformed field $\mathbb{A}^g_\mu$ in components $(A^g)^a_\mu$, we see that the latter transform as follows:

$$\begin{aligned} 2\,\mathrm{tr}\left(\frac{\lambda^a}{2}\mathbb{A}^g_\mu\right) = (A^g)^a_\mu &= A^a_\mu - \frac{i^2}{g}(\partial_\mu \epsilon^a + g A^b_\mu f^{abc} \epsilon^c) \qquad \text{so} \\ (A^g)^a_\mu &= A^a_\mu + \frac{1}{g}\mathcal{D}^{ac}_\mu \epsilon^c \qquad \text{with} \\ \mathcal{D}^{ac}_\mu &= \partial_\mu \delta^{ac} + g A^b_\mu f^{abc} \end{aligned}$$

Now, we rewrite the gauge conditions as follows:

$$\mathcal{F}_b((A^g)^a_\mu(x)) = \underbrace{\mathcal{F}_b(A^a_\mu)}_{=0} + \int d^4y \frac{\delta \mathcal{F}_{b,x}}{\delta A^a_\mu(y)} \frac{1}{g} \mathcal{D}^{ac}_\mu \epsilon^c(y) \tag{19.39}$$

**Note**

$$\mathcal{F}_b(A^a_\mu(x)) \to \mathcal{F}_{b,x}(A^{\cdot}_\mu(\cdot)) = \int d^4y \mathcal{F}_b(A^a_\mu(y))\delta(x-y)$$

is an *operator* in the space of functions of $x, a$, or a *functional* with indices $b, x$.

With the rewriting of $\mathcal{F}_b(A^a_\mu)$ in eq. (19.39), we obtain for $\Delta^{-1}(A^a_\mu(x))$:

$$\Delta^{-1}(A^a_\mu(x)) = \int \prod \mathcal{D}\epsilon^a \prod_{x,b} \delta\left(\int d^4y \frac{\partial \mathcal{F}_b}{\partial A^a_\mu(x)} \delta^4(x-y)\frac{1}{g}\mathcal{D}^{ac}_\mu \epsilon^c(y)\right) \tag{19.40}$$

Since

$$\int d\epsilon_1 \ldots d\epsilon_N \delta(B_{1j_1}\epsilon^{j_1})\ldots\delta(B_{Nj_N}\epsilon^{j_N}) = \int \frac{1}{\det B}\delta^N(\vec{\epsilon})d\epsilon_1\ldots d\epsilon_N = \frac{1}{\det B}$$

eq. (19.40) means that

$$\Delta^{-1}(A^a_\mu(x)) = (\det M)^{-1} \qquad \text{or}$$
$$\boxed{\Delta(A^a_\mu(x)) = \det M} \tag{19.41}$$

with

$$\boxed{M^{xy}_{bc} = \frac{1}{g}\frac{\partial \mathcal{F}_b}{\partial A^a_\mu(x)}\delta(x-y)\mathcal{D}^{ac}_\mu(y)} \tag{19.42}$$

**Exercises**

- In the Lorentz gauge, $\partial^\mu A^a_\mu(x) = 0$, derive the following results for $\mathcal{F}_{b,x}(A^{\cdot}_\mu(\cdot)$:

$$\mathcal{F}_{b,x}(A^{\cdot}_\mu(\cdot)) = \delta^a_b \partial^\mu_{\bar{y}} \, A^a_\mu(\bar{y})\big|_{\bar{y}=x} = \int d\bar{y}\delta^{ba}\partial^\mu_{\bar{y}} A^a_\mu(\bar{y})\delta(\bar{y}-x)$$

  and

$$\frac{\delta \mathcal{F}_{b,x}(A^{\cdot}_\mu(\cdot))}{\delta A^a_\mu(y)} = \delta^{ba}\int d\bar{y}\partial^\mu_{\bar{y}}\delta(\bar{y}-y)\delta(\bar{y}-x) = \delta^{ba}\partial^\mu_x \delta(x-y) = -\,\delta^{ba}\partial^\mu_y\delta(x-y)$$

- Derive the covariant derivative for the adjoint representation: $D_\mu^{ab} = \partial_\mu \delta^{ab} + g f^{abc} A_\mu^c$. In $n \times n$ matrix notation, this is

$$\mathbb{D}_\mu = \partial_\mu \mathbb{1}_{n\times n} - ig[\mathbb{A}_\mu, \cdot]$$

The action on a field in adjoint representation is:

$$\underbrace{2\,\mathrm{tr}(T^b[\mathbb{D}_\mu T^a])}_{D_\mu^{ba}}\, A_\nu^a = \partial_\mu A_\nu^b \underbrace{-2\,i\,g\,\mathrm{tr}(T^b[\mathbb{A}_\mu, \mathbb{A}_\nu])}_{g f^{bac} A_\mu^c A_\nu^a}$$

**Faddeev-Popov trick**

E.g. in the Lorenz gauge,

$$\boxed{\partial^\mu A_{b\mu} = 0} \tag{19.43}$$

before, we obtained the following for $M_{bc}^{xy}$:

$$\begin{aligned} M_{bc}^{xy} = &\frac{1}{g}(-\delta_{ab}\partial_y^\mu \delta(x-y)\mathcal{D}_\mu^{ac}(y)) = \\ &\frac{1}{g}\delta_{ab}\delta(x-y)\partial^\mu \mathcal{D}_\mu^{ac}(y) \end{aligned}$$

To get rid of the $\delta$-functions, we go to a more general gauge, given by

$$\boxed{\partial^\mu A_{b\mu} - B_b(x) = 0} \tag{19.44}$$

which has the same $M$. One can perform a "Gaussian" integral over this $B_b$:

$$\int \mathcal{D}B \exp\left\{ i \int d^4x \left( -\frac{1}{2\xi} B_a(x) B_a(x) \right) \right\} ...\delta(\partial^\mu A_{b\mu} - B_b)$$

This produces a factor

$$\exp\left\{ -\frac{2}{2\xi} i \int d^4x (\partial^\mu A_\mu^a)^2 \right\}$$

in the path integral. Now, we come to the Faddeev-Popov trick: we introduce the Grassmann valued scalar fields $c$ and $\bar{c}$, called *ghost fields*, into the determinant of $M$.

$$\det(iM) = \int \prod_{a,b} \mathcal{D}\bar{c}^a \mathcal{D}c^b \exp\left\{ -i \int dx dy\, \bar{c}^a(x) M_{ab}^{xy} c^b(y) \right\} \tag{19.45}$$

This promotes the det-term to part of the action: the new Lagrangian density is given by

$$\boxed{\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{ferm}}} \tag{19.46}$$

with

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \tag{19.47}$$

$$\mathcal{L}_{\mathrm{ghost}} = -\bar{c}^a \partial^\mu D^{ab}_\mu c^b \tag{19.48}$$

$$\mathcal{L}_{\mathrm{g.f.}} = -\frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 \tag{19.49}$$

$$\mathcal{L}_{\mathrm{ferm.}} = \sum_k \bar{\Psi}^\alpha_{(k)} \left[ \gamma^\mu D_{\mu\alpha}^{\ \beta} - \delta^\beta_\alpha m_{(k)} \right] \Psi_{(k)\beta} \tag{19.50}$$

and

$$\begin{aligned} D^{ab}_\mu &= \delta^{ab} \partial_\mu + g f^{abc} A^c_\mu \quad \text{in adjoint representation} \\ D^\beta_{\mu\alpha} &= \delta^\beta_\alpha - i g A^a_\mu (T^a)^\beta_\alpha \quad \text{in fundamental representation} \end{aligned}$$

**Remarks**

- The parameter $\xi$ varies per gauge. In the Feynman gauge, it is unity; in the Landau gauge, it is zero.
- $(T^a)^\beta_\alpha = (\lambda^a/2)^\beta_\alpha$ is a generator in the fundamental representation; in the adjoint representation, it is $(if^a)^{bc}$.
- In the axial gauge, $e^\mu A^a_\mu = 0$, the ghost Lagrangian density becomes

$$\mathcal{L}_{\mathrm{ghost}} = -\bar{c}^a(x) e^\mu D^{ab}_\mu c^b(x)$$

- In QED, $f^{abc} = 0$; in other words, the ghosts decouple.

# Chapter 20

# Feynman rules in Lorentz covariant gauges

## 20.1 The gluon and ghost propagators

Recall the Lagrangian density derived at the end of chapter 19:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{ferm}}$$

Consider only the quadratic parts of $\mathcal{L}_{\text{YM}}$ and $\mathcal{L}_{\text{g.f.}}$; we will call these parts taken together $\mathcal{L}^{(2)}$. The corresponding action is

$$iS^{(2)} = i\int d^4x \mathcal{L}^{(2)} = i\int d^4x \left\{ \frac{1}{2} A^a_\mu (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu(1-\xi^{-1}))A^a_\nu \right\}$$

where in the second step, we have applied partial integration. The factor between $A^a_\mu$ and $A^a_\nu$ can be written as a matrix $M$, whose Fourier transform is

$$\tilde{M}^{ab}_{\mu\nu} = \left( -k^2 g^{\mu\nu} + k^\mu k^\nu \left( \frac{\xi-1}{\xi} \right) \right) \delta^{ab} \tag{20.1}$$

The inverse of this matrix is the *gluon propagator*:

$$\boxed{(\tilde{M}^{-1})^{ab}_{\mu\nu} = \frac{1}{k^2}\left( -g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2} \right) \delta^{ab} =: -i\Delta_{\mu\nu}(k)} \tag{20.2}$$

- Check:

$$g_{\mu\nu}k^2 - \frac{\xi-1}{\xi}k_\mu k_\nu - (1-\xi)k_\mu k_\nu - \frac{(1-\xi)^2}{\xi}k_\mu k_\nu = g_{\mu\nu}k^2$$

**Note**

Of course, the usual Wick rotation to Euclidean spacetime can be performed:

$$x_0 = -ix_4 \qquad \text{and} \qquad k_0 = ik_4$$

Under this rotation,

$$\begin{aligned} x^2 = x_0^2 - \vec{x}^2 &\to -x_4^2 - \vec{x}^2 = -x_{\mathrm{E}}^2 \\ \partial^2 &\to -\partial_{\mathrm{E}}^2 \\ g^{\mu\nu} &\to -\delta^{\mu\nu} \\ A_0 &\to -iA_4 \\ \mathcal{L} &\to -\mathcal{L}_{\mathrm{E}} \\ e^{iS} &\to e^{-S_{\mathrm{E}}} \end{aligned}$$

These minus signs are such that the action in Euclidean spacetime,

$$S_{\mathrm{E}} = \int d^4x_{\mathrm{E}} \mathcal{L}_{\mathrm{E}}$$

is positive definite. The quadratic part discussed above becomes

$$S_{\mathrm{E}}^{(2)} = \frac{1}{2} \int d^4x_{\mathrm{E}} A_\mu^a(x) \underbrace{\left(-\delta_{\mu\nu}\partial_{\mathrm{E}}^2 + \partial_{\mathrm{E}}^\mu \partial_{\mathrm{E}}^\nu \left(1 - \frac{1}{\xi}\right)\right)}_{k_{\mathrm{E}}^2\delta_{\mu\nu} - k_{\mathrm{E}\mu}k_{\mathrm{E}\nu}(1-\frac{1}{\xi})} A_\nu^a(x) \tag{20.3}$$

The factor between brackets is positive definite for $\xi > 0$. For source terms, the transformation to the Euclidean looks like this:

$$\exp\left\{i \int d^4x j^{a\mu}(x) A_\mu^a(x)\right\} \to \exp\left\{- \int d^4x_{\mathrm{E}} (j^{\mathrm{E}})_\mu^a(x) (A^{\mathrm{E}})_\mu^a(x)\right\}$$

As usual, we obtain the propagator from the generating functional by partial differentiation:

$$\langle 0| \mathrm{T}(A_\mu^a(x) A_\nu^b(y)) |0\rangle = \int \mathcal{D}A A_\mu^a(x) A_\nu^b(y) e^{iS} = \tag{20.4}$$

$$\frac{\delta}{i\delta j^{a\mu}(x)} \frac{\delta}{i\delta j^{b\nu}(y)} \int \mathcal{D}A \exp\left\{iS + i \int d^4x j^{a\mu} A_\mu^a(x)\right\}\Bigg|_{j=0} \tag{20.5}$$

For completeness' sake, we also give the quark propagator:

$$\langle 0| \mathrm{T}(\Psi_{j\alpha}(x) \bar{\Psi}_{j\beta}(y)) |0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \tilde{S}_{\mathrm{F}}^{(j)}(p)$$

With the familiar relation $(i\gamma^\mu\partial_\mu - m)S_{\mathrm{F}}/i = \delta^4(x-y)$, we get

$$\tilde{S}_{\mathrm{F}}^{(j)} = \frac{i}{\not{p} - m_j} = \frac{i(\not{p} + m_j)}{p^2 - m^2 + i\epsilon} \tag{20.6}$$

### Ghost propagator

To conclude this section, the ghost propagator:

$$\langle 0|\, \mathrm{T}(c^a(x)\bar{c}^b(y))\,|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} D^{ab}(p^2) \tag{20.7}$$

Note that $c^a$ and $\bar{c}^b$ are not each other's complex conjugate, but separate fields. The $D^{ab}$ in the integral is given by

$$D^{ab}(p^2) = \frac{i}{p^2 + i\epsilon}\delta^{ab} \tag{20.8}$$

This is the same as for a scalar field, except that the ghosts are fermionic, and hence give rise to an extra minus sign in the case of closed loops.

## 20.2 Vertices

We can read off the vertices from the action, after applying partial integration. The action is

$$iS = \frac{i}{4}\int d^4x(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)^2$$

The 3-gluon vertex (note that although we use the word 'gluon', which is specific to QCD, what we do here is general) then is

$$\begin{aligned}
&\frac{-i}{2} g f_{abc}\int d^4x(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu} = \\
&\frac{-i}{2} g f_{abc}\int d^4x \left[(A^{c\nu}\partial_\mu A^a_\nu)A^{b\mu} - (A^{b\mu}\partial_\nu A^a_\mu)A^{c\nu}\right] = \\
&\frac{-i}{2} g f_{abc}\int d^4x \left[(A^{c\nu}\overleftrightarrow{\partial}_\mu A^a_\nu)A^{b\mu}\right] = \\
&\underbrace{\frac{1}{3}\frac{-i}{2}}_{\frac{-ig}{3!}} g f_{abc}\int d^4x \left\{\left[A^{a\mu}(A^{b\nu}\overleftrightarrow{\partial}_\mu A^c_\nu)\right] + \text{cyclic permutations in } (a,\, b,\, c)\right\}
\end{aligned}$$

Note:

$$\begin{pmatrix} a & b & c \\ \mu & \nu & \rho \end{pmatrix}$$

is symmetric under exchange of $a$, $\mu \leftrightarrow b$, $\nu$, i.e., they are bosons. All of this gives the Feynman diagram shown in figure 20.1.

The Fourier transformed version of this expression is given by

$$gf^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^{mu} + g^{\rho\mu}(q-k)^\nu\right]$$

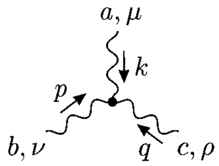



Figure 20.1: 3-gluon vertex

For a 4-vertex, the idea is similar:

$$\frac{-i}{4}g^2 f_{abc}A^b_\mu A^c_\nu f_{ade}A^{d\mu}A^{e\nu}$$

is to be symmetrized.

**Propagators and vertices for QCD**

**Propagators**

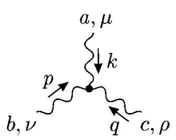
gluon  $\frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}(1-\xi)\right)\delta_{ab}$

ghost 

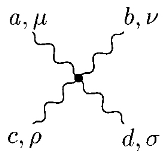
$\frac{i}{k^2+i\epsilon}\delta_{ab}$

fermion  $\frac{i}{(\gamma p-m+i\epsilon)_{\beta\alpha}}$

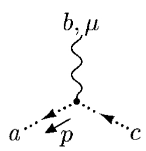
**Vertices**



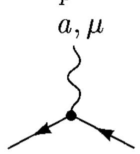

$$gf^{abc}[g^{\mu\nu}(k-p)^\rho+g^{\nu\rho}(p-q)^\mu+g^{\rho\mu}(q-k)^\nu]$$




$$\begin{aligned}-ig^2[&f^{abc}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})+\\&f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})+\\&f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})]\end{aligned}$$




$$gf^{abc}p^\mu$$




$$ig\gamma^\mu\frac{\lambda_a}{2}$$

To obtain the expression for a ghost-gluon-vertex, one has to integrate partially; the result is

$$-i\int \bar{c}^a\partial^\mu(gf^{abc}A^c_\mu)c^b = i\int \partial^\mu\bar{c}^a(gf^{abc}A^c_\mu)c^b$$

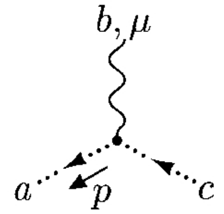



Figure 20.2: Ghost-gluon vertex

Including statistical factors and minus signs for closed fermion loops, this completes the Feynman rules for QCD, a non-abelian gauge theory. We now have derivations of these rules for both abelian and non-abelian gauge theories, and can simply start calculating graph expressions, and do regularization and renormalization after fixing a gauge.

## 20.3 BRST-symmetry

Having fixed the gauge, we would like to retain some of the gauge symmetry, e.g. in order to obtain equations like the Ward identity. Introducing an auxiliary field $B^a$, the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{2}G^a_{\mu\nu}G^{a\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi - \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu + \bar{c}^a(-\partial^\mu D^{ac}_\mu)c^c$$

(The factor of $1/g$ before the $\bar{c}\ldots c$-term has been defined away.) It can easily be checked that after integration over this auxiliary field, $\int \mathcal{D}B^a$, one obtains the original form. This is only possible because we have a squared term $(\partial_\mu A^\mu)^2$. This Lagrangian has a symmetry with a Grassmann valued *global* parameter $\varepsilon$ (this parameter belongs to an infinitesimal transformation), under a *supertransformation*:

$$\bar{\delta}_B A^a_\mu = \varepsilon D^{ac}_\mu c^c, \tag{20.9}$$

$$\bar{\delta}_B \Psi = ig\varepsilon c^a T^a \Psi, \tag{20.10}$$

$$\bar{\delta}_B c^a = -\frac{1}{2}g\varepsilon f^{abc}c^b c^c, \tag{20.11}$$

$$\bar{\delta}_B \bar{c}^a = \varepsilon B^a \qquad \text{and} \qquad \bar{\delta}_B B^a = 0 \tag{20.12}$$

The first two of these are a gauge transformation with $\varphi^a(x) = g\varepsilon c^a(x)$. The fact that this transformation indeed leaves the Lagrangian invariant can be seen as follows:

$$\bar{\delta}_B(D_\mu^{ac}c^c) = D_\mu^{ac}\left(\frac{-1}{2}g\varepsilon f^{cde}c^d c^e\right) + (gf^{ace}\varepsilon D_\mu^{ed}c^d)c^c =$$
$$-\frac{1}{2}g^2\varepsilon f^{abc}f^{cde}(-A_\mu^b c^d c^e + A_\mu^d c^e c^b + A_\mu^e c^b c^d) = 0$$

where in the last step, we have observed that the indices $b$, $d$ and $e$ are cyclically permuted, and used the Jacobi identity

$$[T^a, [T^b, T^c]] + \text{cyclic permutations} = 0$$

The $A_\mu^a$-transformation in the fourth term is cancelled by the $\bar{c}^a$-transformation in the last one.

In general, one writes

$$\boxed{\bar{\delta}_B\Phi = \varepsilon Q\Phi} = \varepsilon\delta_B\Phi \tag{20.13}$$

or $\varepsilon[Q, \Phi]_\pm$ after quantization, with the minus for bosonic, and the plus for fermionic $\Phi$. Here, $Q$ is the generator of these BRST-transformations, and $\Phi = A_\mu^a, \Psi, c^a, \bar{c}^a, B^a$. The transformation is called *nilpotent*:

$$\delta_B^2 = 0, \qquad Q^2 = 0$$

(from now on, we will write $\delta_B$ without the $\varepsilon$.) This implies that $Q$ is also Grassmann valued. In the case of, e.g. bosonic $\Phi$, one has

$$Q(Q\Phi - \Phi Q) + (Q\Phi - \Phi Q)Q = [Q^2, \Phi]$$

Note: the way it is presented here, it might look as though BRST-symmetry were specific to certain gauges, but it is, in fact, completely gauge independent (see Kugo's book for more on this).

There is another way of deriving eqs. (20.9) - (20.12): one postulates

$$\delta_B^2 = 0$$

Then, eq. (20.10) follows immediately from eq. (20.9). *Defining* $\bar{c}$ by

$$\delta_B\bar{c}^a(x) = B^a$$

one has eq. (20.11); $B^a$ is the unspecified auxiliary field, for which, using $\delta_B^2 = 0$, eq. (20.12) holds:

$$\delta_B B^a = \delta_B^2\bar{c}^a(x) = 0$$

Then, a constraint, an arbitrary gauge $F^a(A,\Psi,B)=0$, is used to *define*

$$\mathcal{L}_{\text{g.f.}}+\mathcal{L}_{\text{ghost}}=\delta_B(\bar{c}^aF^a) \tag{20.14}$$

Taking e.g. $F^a=\partial^\mu A^a_\mu-\xi B^a/2$, this gives

$$\begin{aligned}\mathcal{L}_{\text{g.f.}}+\mathcal{L}_{\text{ghost}} = \delta_B\left[\bar{c}^a\left(\partial^\mu A^a_\mu-\frac{\xi}{2}B^a\right)\right] &= \\ B^a\partial^\mu A^a_\mu-\frac{\xi}{2}B^aB^a-\bar{c}^a\partial_\mu D^{ac}_\mu c^c&\end{aligned}$$

Now performing a path integral over $B^a$ to remove the auxiliary field gives the usual (covariant gauge) Lagrangian with $-(\partial_\mu A^a_\mu)^2/2\xi$. The BRST Lagrangian does not require any further constraints for quantization and can be canonically quantized (again, see Kugo for more details).

The nilpotent operator $Q$ allows for a (unique) *cohomology* decomposition:

$$\boxed{|\Psi\rangle=|\Psi_0\rangle+|\Psi_1\rangle+|\Psi_2\rangle} \tag{20.15}$$

$Q$ acts on these $|\Psi_0\rangle$, $|\Psi_1\rangle$ and $|\Psi_2\rangle$ as follows:

- $Q\,|\Psi_0\rangle=0, \qquad$ but $\qquad |\Psi_0\rangle\neq Q\,|\Psi_1'\rangle$
- $|\Psi_2\rangle=Q\,|\Psi_1'\rangle$
- $Q\,|\Psi_1\rangle\neq 0$

This means that $|\Psi_0\rangle$ is closed, but not exact; $|\Psi_2\rangle$ is exact, and hence closed as well, and $|\Psi_1\rangle$ is not closed, and not exact either (and orthogonal to $|\Psi_0\rangle$ and $|\Psi_2\rangle$). The whole theory can be projected onto the subspace of $|\Psi_0\rangle$ - this is the *physical* subspace - and the $S$-matrix is *unitary* in this subspace. The rest is perpendicular: $\langle\Psi_2|\Psi_0\rangle=\langle\Psi_1|\,Q\,|\Psi_0\rangle=0$. The $\langle\Psi_2|$-space has a *null* inner product: $\langle\Psi_2|\Psi_2'\rangle=\langle\Psi_1|\,Q\,|\Psi_2'\rangle=\langle\Psi_1|\,Q^2\,|\Psi_1'\rangle=0$. This is becoming a bit technical, however; one can also simply take the limit where the gauge coupling $g\to 0$, and consider the following Fourier transforms:

$$\begin{aligned}&(i)\quad \delta\tilde{A}^a_\mu=-i\varepsilon k_\mu\tilde{c}^a\\ &(ii)\quad \delta\tilde{c}^a=0\\ &(iii)\quad \delta\tilde{\bar{c}}^a=\varepsilon\tilde{B}^a=-i\varepsilon\xi k^\mu\tilde{A}^a_\mu\end{aligned}$$

Here, in $(iii)$, $\tilde{B}^a$ has quanta with $k^\mu\varepsilon_\mu\neq 0$ (they are *polarized backwards*) and is obtained by a $Q$-transform of $\tilde{\bar{c}}$; $\tilde{c}$ is obtained by a $Q$-transform of $\tilde{A}^a_\mu$

in $(i)$, and $\tilde{A}^a_{\mu\text{transverse}}$ is annihilated by $Q$. This implies

$$\begin{aligned} \tilde{c}^a,\, \tilde{A}^a_\mu \propto \begin{pmatrix} k_0 \\ -\vec{k} \end{pmatrix} &\in \mathfrak{H}_2 & \\ \tilde{A}^a_{\mu\text{transverse}} &\in \mathfrak{H}_0 & (\epsilon^\mu_\perp \delta \tilde{A}^a_\mu = 0) \\ \tilde{\tilde{c}}^a,\, \tilde{A}^a_\mu \propto k_\mu &\in \mathfrak{H}_1 & (\delta \tilde{\tilde{c}}^a,\, \delta A^a_{\mu\,\text{elgm}} \neq 0) \end{aligned}$$

## 20.4 Quantizing under constraints

In gauge theories, vector fields related by gauge transformations represent the same physical situation. As discussed above, we can fix a gauge by introducing constraints

$$\mathcal{F}_b(A^a_\mu) = 0$$

This restricts the gauge orbits to single points. Then, one has to quantize a system with constraints. In mechanics, systems with constraints are well-known; in QM, this is less standard in lectures. In these lecture notes, we will quantize on the basis of the path integral formulations, but before we can go into that discussion, a few remarks on the canonical formalism are appropriate.

The equations for the canonical momenta

$$p_i = \frac{\partial L}{\partial(\partial_t q_i)}(q, \dot{q}) \tag{20.16}$$

may not be solvable. In order for a solution to exist, the $\dot{q}_i$ that are needed to write an expression for $H$ have to satisfy the criterium of invertibility:

$$\det\left(\frac{\partial^2 L(q,\dot{q})}{\partial(\partial_t q_i \partial_t q_j)}\right) \neq 0 \tag{20.17}$$

If the equations (20.16) for the $p_i$ are not independent, i.e. not solvable, there exists a set of so-called *primary constrains*

$$\begin{aligned} \phi_A(q,p) = 0 \qquad A = 1, \ldots, M \leq N \qquad \text{with} \\ M = N - \operatorname{rank}\left(\frac{\partial^2 L(q,\dot{q})}{\partial(\partial_t q_i \partial_t q_j)}\right) \end{aligned} \tag{20.18}$$

To obtain a Hamiltonian for the constrained system, let us start with the well-known expression $H = \dot{q}_i p_i - L(q, \dot{q})$, which is for an unconstrained system. Consider the variation of $H$:

$$\delta H = \dot{q}_i \delta p_i + \underbrace{p_i \delta \dot{q}_i - \frac{\partial L}{\partial(\partial_t q_i)} \delta \dot{q}_i}_{=0} - \frac{\partial L}{\partial q_i} \delta q_i =$$

$$\delta p_i \dot{q}_i - \frac{\partial L}{\partial q_i} \delta q_i \stackrel{!}{=} \delta p_i \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial q_i} \delta q_i$$

Since eqs. (20.16) are not independent, neither are $\delta p_i$ and $\delta q_i$, and from eq. (20.18) we get the constraint

$$\delta p_i \frac{\partial \phi_A}{\partial p_i} + \frac{\partial \phi_A}{\partial q_i} \delta q_i = 0$$

Introducing this constraint into our system of equations by means of Lagrange multipliers $\lambda^A$, i.e. writing $H + \lambda^A \phi_A$, we get

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + \frac{\partial \phi_A}{\partial p_i} \lambda^A \tag{20.19}$$

$$-\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i} + \frac{\partial \phi_A}{\partial q_i} \lambda^A \tag{20.20}$$

Now, using eqs (20.16), the Euler-Lagrange equations of motion, and eq. (20.20) successively, we get

$$\dot{p}_i = \frac{d}{dt} \frac{\partial L}{\partial(\partial_t q_i)} = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} - \frac{\partial \phi_A}{\partial q_i} \lambda^A$$

This can also be written with the help of the Poisson bracket:

$$\{F, G\}_{\mathrm{P}} = \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \tag{20.21}$$

This yields, for eqs. (20.19) and (20.20),

$$\dot{q}_i = \{q_i, H\}_{\mathrm{P}} + \{q_i, \phi_A\}_{\mathrm{P}} \lambda^A$$
$$\dot{p}_i = \{p_i, H\}_{\mathrm{P}} + \{p_i, \phi_A\}_{\mathrm{P}} \lambda^A$$

In general, for arbitrary functions $F(p, q)$, one has

$$\dot{F}(p, q) = \left\{F, H + \lambda^A \phi_A\right\}_{\mathrm{P}} - \underbrace{\left\{F, \lambda^A\right\}_{\mathrm{P}} \phi_A}_{=0 \text{ for } \phi_A = 0}$$

or, in the submanifold $\Gamma^*$ defined by $\phi_A = 0$,

$$\boxed{\dot{F} = \left\{F, \tilde{H}\right\}_{\mathrm{P}}} \tag{20.22}$$

where $\tilde{H} = H + \lambda^A \phi_A$.

### 20.4.1 Stability of constraints

We want our constraints to be stable, i.e. time independent:

$$\dot{\phi}_A = \{\phi_A, H\}_\mathrm{P} + \{\phi_A, \phi_B\}_\mathrm{P} \lambda^B \stackrel{!}{=} 0 \tag{20.23}$$

Rewriting the constraints $\phi_A$ into suitable linear combinations of each other, we can write

$$\{\phi_A, \phi_B\}_\mathrm{P} = C_{AB} = \begin{pmatrix} C_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \tag{20.24}$$

For the index $A$, we can write $A = \alpha, a$, with

$$1 \leq \alpha, \beta, \leq r_1 \qquad \text{and} \qquad r_1 + 1 \leq a, b, \leq M$$

This allows us to distinguish two subspaces:

(i) the $\alpha, \beta$-subspace, where we can solve for $\lambda^B$ and thus fulfill the stability equation (20.23)

(ii) the remaining space, where we have to postulate $\{\phi_A, H\}_\mathrm{P} = 0$

The condition required in subspace (ii) can be fulfilled if

$$\{\phi_A, H\}_\mathrm{P} = C_a^B \phi_B = 0$$

Otherwise, we have to introduce so-called *secondary constraints*, add these to the original set, and go through the entire procedure again (note that the only difference between secondary and primary constraints is the time at which they are introduced into the problem; they are not qualitatively different). Having done this often enough, we find the Hamiltonian for the constrained system:

$$\boxed{\tilde{H} = H - \phi_\alpha (C^{-1})^{\alpha\beta} \{\phi_\beta, H\}_\mathrm{P} + \phi_a \lambda^a} \tag{20.25}$$

Constraints $\phi^*$ with

$$\boxed{\{\phi^*, \phi_A\}_\mathrm{P} = 0 \quad \forall A}$$

are called *first class* constraints; all others are *second class*. If all constraints are second class,

$$\{\phi_A, \phi_B\}_\mathrm{P} = C_{\alpha\beta}$$

so $C^{-1}$ exists.

With eq. (20.25) in mind, we define the *Dirac bracket*:

$$\boxed{\{F, G\}_\mathrm{D} := \{F, G\}_\mathrm{P} - \{F, \phi_\alpha\}_\mathrm{P} (C^{-1})^{\alpha\beta} \{\phi_\beta, G\}_\mathrm{P}} \tag{20.26}$$

This has the property that

$$\boxed{\dot{F} = \{F, \tilde{H}\}_{\mathrm{P}} = \{F, H\}_{\mathrm{D}}}$$

In the constrained space, the submanifold $\Gamma^*$, the Dirac and Poisson brackets coincide. This becomes particularly obvious if one writes them out in coordinates that fulfill the constraints.

The quantization procedure for the constrained system is to substitute the (anti-)commutator for the Dirac bracket:

$$\boxed{i\{F, G\}_{\mathrm{D}} \to [F, G]}$$

In the path integral formulation, nothing changes if the path integral is written in constrained coordinates (i.e. fields). However, this is impractical, since it would require one to solve all the constraint equations. Instead, one may use the following, quicker, formulation:

$$\begin{aligned} T = \langle \Psi_F, t_F | \Psi_I, t_I \rangle = \\ \int \mathcal{D}p\mathcal{D}q\Psi_F^*\Psi_I \prod_t \left( \prod_{\alpha=1}^{2m} \delta(\phi_\alpha[\det(\{\phi_\alpha, \phi_\beta\}_{\mathrm{P}})]^{1/2} \right) \times \\ \exp\left\{ i \int dt[p\dot{q} - H(p, q)] \right\} \end{aligned} \tag{20.27}$$

(The proof will not be given here; again, see Kugo for details.) Performing the $\mathcal{D}p$-integral, one can arrive at the path integral for gauge fields discussed before. Gauge conditions are second class constraints.

# Chapter 21

# Renormalization in gauge theories

## 21.1 Perturbative expansion in non-abelian gauge theory

Let us start our discussion of renormalization at the expansion. We will concentrate on QCD, this being the most common example of a non-abelian gauge theory. It turns out that we can classify Feynman graphs for QCD like we could do for $\Phi^4$-theory. We have a renormalizable theory in 4 dimensions, but since we will use dimensional regularization, we will work in $D$ dimensions.

Like before, we define the *superficial degree of divergence* $\omega$:

$$\boxed{\omega(\Gamma) = D \cdot L + V_3 + V_{\mathrm{F.P}} - 2I_A - I_\Psi - 2I_{\mathrm{F.P.}} - \frac{1}{2}E_{F.P.}} \tag{21.1}$$

$L$ is the number of loops; $V_3$ the number of 3-vertices; $V_{\mathrm{F.P.}}$ the number of Fadeev-Popov ghost vertices; $I_A$ the inner gauge propagators; $I_\Psi$ the inner fermion propagators; $I_{\mathrm{F.P.}}$ the ghost propagators and $E_{\mathrm{F.P.}}$ the outer Fadeev-Popov fields. For truncated 1PI graphs, these quantities have the following relations:

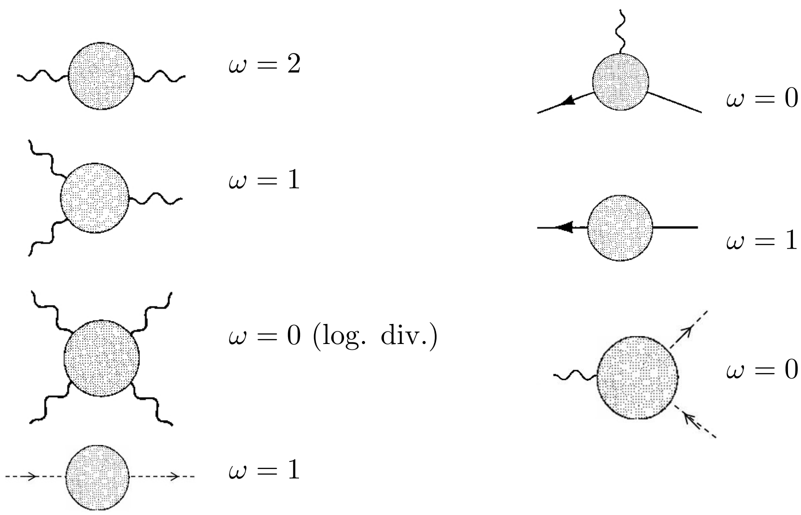
$$\begin{aligned}
2V_{\Psi A} &= 2I_\Psi + E_\Psi \\
2V_{\mathrm{F.P.}} &= 2I_{\mathrm{F.P.}} + E_{\mathrm{F.P.}} \\
3V_3 + 4V_4 + V_{\Psi A} + V_{\mathrm{F.P.}} &= 2I_A + E_A \\
L - 1 &= \sum_i I_i - \sum_k V_k
\end{aligned} \tag{21.2}$$

Using these relations, we can rewrite $\omega(\Gamma)$:

$$\boxed{\omega(\Gamma) = 4 - E_A - \frac{3}{2}E_\Psi + \left(-\frac{3}{2}E_{\mathrm{F.P.}}\right) - (4 - D)L} \tag{21.3}$$

Note that for $D = 4$, this depends only on the outer lines; this is typical for renormalizable theories.

For $D = 4$, the following graphs are superficially divergent:




Note: for $D < 2$, $\omega < 0$ for all diagrams, i.e. this is a convergent theory in less than 2 dimensions.

The above superfically divergent graphs, the renormalization pieces, correspond exactly to the terms in the tree-level Lagrangian - as they should in a renormalizable theory. Now, one has to implement the same procedure as for $\Phi^4$-theory: decompose graphs into skeleton graphs, iterate this process, and be convinced that the renormalization program works.

Wave function renormalizations ($Z_2$, $Z_3$ and $\tilde{Z}_3$), gauge coupling renormalizations ($Z_1$, $Z_4$, $\tilde{Z}_1$, $Z_{1F}$) and mass renormalization (fermions) produce counterterms. The gauge coupling renormalizations are not independent if the theory arises from a $\mathcal{L}_0$ gauge theory. The Lagrangian $\mathcal{L}_0$ takes the form

$$\begin{aligned}
\mathcal{L}_0 = \mathcal{L} + \delta\mathcal{L}_{\mathrm{ct}} = & -\frac{1}{4} Z_3 (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) - \\
& - \frac{1}{2\alpha}(\partial_\mu A^{\mu a})^2 + g Z_1 f_{abc} A_\mu^a A_\nu^b \partial^\mu A^{\nu c} - \\
& - \frac{g^2}{4} Z_4 f_{abc} f_{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} + \\
& \tilde{Z}_3 \partial_\mu \bar{c}^a \partial^\mu c^a + g \tilde{Z}_1 (\partial_\mu \bar{c}^b A^{\mu a} c^c f_{abc}) + \\
& Z_2 \bar{\Psi}(i\not{\partial} - m)\Psi - Z_2 \delta_m \bar{\Psi}\Psi - i g Z_{1F} \bar{\Psi} \not{A}_a \frac{\lambda^a}{2} \Psi
\end{aligned} \tag{21.4}$$

with

$$\mathbb{A}_0 = Z_3^{1/2}\mathbb{A}; \quad c_0 = \tilde{Z}_3^{1/2}c; \quad \Psi_0 = Z_2^{1/2}\Psi; \quad \alpha_0 = Z_3\alpha; \quad m_0 = m + \delta m$$
$$g_0 = Z_1 Z_3^{-3/2} = (Z_4 Z_3^{-2})^{1/2} g = \tilde{Z}_1 \tilde{Z}_3^{-1} \tilde{Z}_3^{-1/2} g = Z_{1F} Z_3^{-1/2} Z_2^{-1} g \tag{21.5}$$

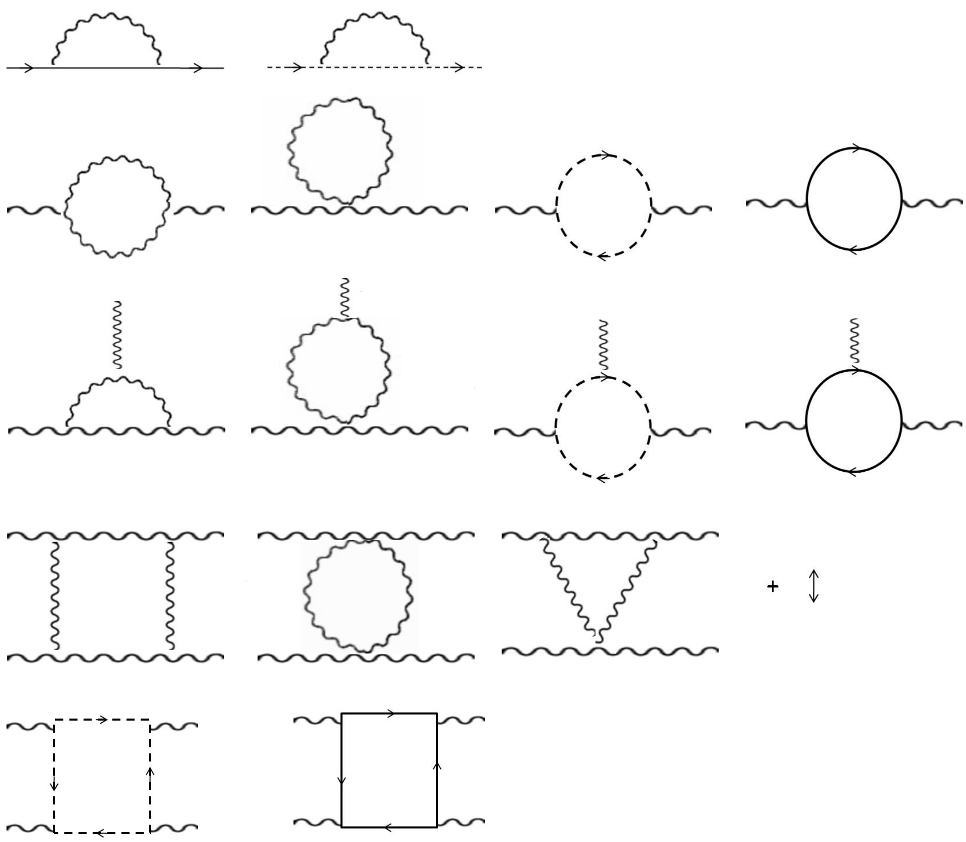
we can derive the following relations between the renormalization factors:

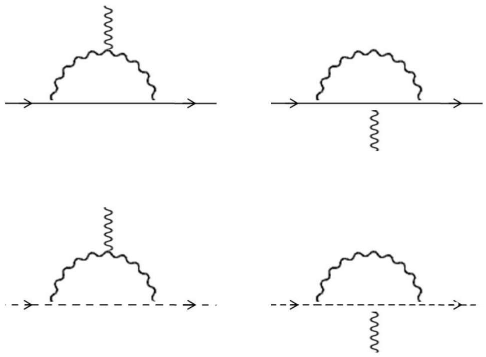
$$\frac{Z_4}{Z_1} = \frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2} \tag{21.6}$$

This gives $\mathcal{L}_0(\mathbb{A}_0, c_0, \Psi_0, g_0, \alpha_0, m_0)$ in the usual normalization. The counterterms $\delta\mathcal{L}_{c+}$ can be derived from the above by splitting off $\mathcal{L}_0(\mathbb{A}, c, \Psi, g, \alpha, m)$. In dimensional regularization, $g \to g\mu^\epsilon$.

### 21.1.1 1-loop graphs

We have the following 1PI 1-loop graphs:





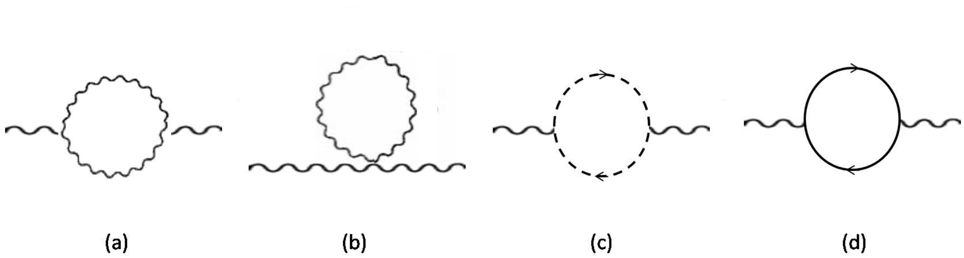
### Note

Dimensional regularization functions like for $\Phi^4$-theory. In the Euclidean notation, $\gamma_0 \to i\gamma_4$ for Dirac fermions, and the $\gamma$-matrices have the following relations in D dimensions:

$$\begin{aligned}
\gamma_\mu\gamma_\nu &= -\,D\mathbb{1}_D; \qquad \gamma_\mu\gamma_\rho\gamma_\mu = (D-2)\gamma_\rho; \qquad \{\gamma_\mu,\gamma_\nu\} = -2\delta_{\mu\nu};\\
\mathrm{tr}\gamma_\mu\gamma_\nu &= -\,D\delta_{\mu\nu}; \qquad \gamma_\rho\gamma_\mu\gamma_\nu\gamma_\rho = -(D-4)\gamma_\mu\gamma_\nu + 4\delta_{\mu\nu};\\
\gamma_\rho\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho &= (6-D)\gamma_\sigma\gamma_\nu\gamma_\mu - 2(D-4)(\delta_{\mu\nu}\gamma_\sigma - \delta_{\mu\sigma}\gamma_\nu - \delta_{\nu\sigma}\gamma_\mu)
\end{aligned}$$

### Vacuum polarization

The following graphs correspond to vacuum polarization:




These are all quadratically divergent in four dimensions:

- The integral corresponding to graph (b) is

$$\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{1}{l^2+m^2} = \frac{\Gamma(1-\omega)}{(4\pi)^2\Gamma(1)}\frac{1}{(0+m^2)^{(1-\omega)}}$$

  where the last fraction goes to zero as $m^2 \to 0$.

- Graph (d) can be kept separate (in that case, we are simply considering the theory without fermions).

- Computing the total vacuum polarization in the Feynman gauge ($\xi = 1$) and leaving out the fermions, we add graphs (a) and (c); (b) is zero in the Feynman gauge.

$$\begin{aligned}\text{(a)} + \text{(c)} = \Pi^{ab}_{\mu\nu} = \frac{g^2}{32\pi^2} f^{acd} f^{cdb} (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \times \\ \left( \frac{10}{3} \left( \frac{1}{\epsilon} - \gamma + \log 4\pi - \log \frac{p^2}{\mu^2} \right) + \frac{62}{9} \right)\end{aligned} \tag{21.7}$$

Note: in general "covariant" gauges, this remains $\xi$-indepedent. The momentum dimensions of the relevant quantities are as follows:

$$\begin{aligned} \dim \mathcal{L} = 2\omega; \qquad & \dim A = \omega - 1; \qquad \dim g^2 A^4 = 2\omega \qquad \text{so} \\ & \dim g = 2 - \omega \end{aligned}$$

In dimensional regularization, we want to keep $g$ dimensionless and substitute it by $g\mu^{2-\omega}$. Inspecting eq. (21.7), we see that only the transversal part of the gluon propagator is involved. The general form of the 2-point function is:

$$\int d^4x e^{ipx} \langle 0 | \, \mathrm{T}(A^a_\mu(x) A^b_\nu(0) \, | 0 \rangle = \left( \delta_{\mu\nu} \frac{p^2 - p_\mu p_\nu}{p^2} \right) A^{ab}_T(p^2) + p^\mu p^\nu A^{ab}_L(p^2)$$

To zeroth order,

$$\begin{aligned} A^{ab}_T &= \delta^{ab} \frac{1}{p^2} \qquad \text{transversal} \\ A^{ab}_L &= \delta^{ab} \xi \frac{1}{p^4} \qquad \text{longitudinal} \end{aligned}$$

Since $p_\mu(p^2 \delta_{\mu\nu} - p_\mu p_\nu) = 0$, we can define the *projection matrix* $P$ ($P^2 = P$):

$$P_{\mu\nu} = \frac{\delta_{\mu\nu} p^2 - p_\mu p_\nu}{p^2} \tag{21.8}$$

With this, we can write the full propagator:

$$i \left( \frac{P_{\mu\nu}}{p^2} + \xi \frac{p_\mu p_\nu}{p^4} \right) + i^2 \left( \frac{P_{\mu\mu'}}{p^2} + \xi \frac{p_\mu p_{\mu'}}{p^4} \right) \frac{\Pi_{\mu'\nu'}}{i} \left( \frac{P_{\nu\nu'}}{p^2} + \xi \frac{p_\nu p_{\nu'}}{p^4} \right) + \dots$$

where $\Pi_{\mu\nu} = P_{\mu\nu} p^2 \Pi(p^2)$; this can be rewritten as

$$\begin{aligned} iP_{\mu\nu} \left( \frac{1}{p^2} + \frac{\Pi^{(\text{ren})}}{p^2} + \frac{\Pi^{(\text{ren})2}}{p^2} + \dots \right) + i\xi \frac{p_\mu p_\nu}{p^4} = \\ iP_{\mu\nu} \left( p^2 (1 - \Pi^{(\text{ren})}))^{-1} \right) + i\xi \frac{p_\mu p_\nu}{p^4} \end{aligned}$$

This propagator still has a pole at $p^2 = 0$, which implies that the gluon remains massless.

In order to avoid IR problems we renormalize (with factor $Z_3$) at $p^2_{\text{Eucl}} = \mu^2$ (i.e., $p^2_{\text{Mink}} = -\mu^2$); then,

$$\Pi^{(\text{ren})}(\mu^2) = 0 \tag{21.9}$$

by our choice of the counterterm .

Note: the $\mu$ is a natural renormalization scale in dimensional regularisation.

For graph (d), the expression is

$$\Pi^{ab\,(\text{ferm})}_{\mu\nu} = \frac{-g^2}{16\pi^2} n_f \operatorname{tr}\left(\frac{\lambda^a\lambda^b}{4}\right)(\delta_{\mu\nu}p^2 - p_\mu p_\nu)\frac{4}{3}\left(\frac{1}{\epsilon}\dots\right)$$

where $n_f$ is the number of Dirac fermions in the fundamental representation. The gauge boson contribution (a) + (c) has the opposite sign.

For the generators, we have the following relation:

$$T^aT^a = \frac{N}{d}C(r)\mathbb{1}$$

with the Dynkin index $C$ and the Casimir operator $C_2(r)$:

$$C_2(r) = \frac{N}{d}C(r) \qquad \operatorname{tr}T^aT^b = \delta^{ab}C(r)$$

in some representation $r$. Here, $N$ is the dimension of the group algebra (e.g. $n^2-1$ for $SU(n)$), and $d$ the dimension of the representation. Thus, $C_{\text{fund}} = \frac{1}{2}$, and $C_{\text{adjoint}} = n$.

Similarly, one can evaluate the couplings (there are relations between these; see e.g. eq. (21.6)).

A particularly elegant approach is the background field method (see e.g. Peskin & Schröder):

$$A^a_\mu = A^{a\,\text{BG}}_\mu + A^{a\,\text{quantum}}_\mu$$

The gauge invariance of the background field effective action becomes manifest. Consistently, one obtains

$$\begin{aligned}
Z_{1_f} &= 1 - \frac{g^2}{16\pi^2}\left[C_{\text{adj}} + C_{\text{fund}}\frac{N}{d_f}\right]\left(\frac{1}{\epsilon}+\dots\right)\\
Z_2 &= 1 - \frac{g^2}{16\pi^2}\left[C_{\text{fund}}\frac{N}{d_f}\right]\left(\frac{1}{\epsilon}+\dots\right)\\
Z_3 &= 1 - \frac{g^2}{16\pi^2}\left[\frac{5}{3}C_{\text{adj}} - \frac{4}{3}C_{\text{fund}}\right]\left(\frac{1}{\epsilon}+\dots\right)
\end{aligned}$$

$g_0$ is independent of $\mu$, so $g$ is a function of $\mu$: $g(\mu)$. This is called a 'running coupling'; it is no longer constant:

$$\begin{aligned} g_0 = g\mu^\epsilon \frac{Z_{1\mathrm{F}}}{Z_2 Z_3^{1/2}} = g\mu^\epsilon Z_1 Z_3^{-3/2} = \cdots = \\ g\mu^\epsilon \left[ 1 - \frac{g^2}{16\pi^2}\left(\frac{11}{6}n - \frac{2}{3}\frac{1}{2}n_f\right)\left(\frac{1}{\epsilon} + \text{finite}\right)\right] \end{aligned} \tag{21.10}$$

The $\beta$-function can be calculated just from $\Pi$.

$$\begin{aligned} \mu\frac{dg_0}{d\mu} = 0 = \Bigg[ \mu\frac{dg}{d\mu} + g\epsilon - \frac{g^3}{16\pi^2}\left(\frac{11}{6}n - \frac{2}{3}\cdot\frac{1}{2}n_f\right) \\ - 3g^2\mu\frac{dg}{d\mu}\left(\frac{11}{6}n - \frac{2}{3}\cdot\frac{1}{2}n_f\right)\frac{1}{\epsilon}\Bigg]\mu^\epsilon \end{aligned}$$

To lowest order in $g$, only the first two terms in the the square brackets contribute, which allows one to calculate $dg/d\mu$ in the last term. So, to order $\epsilon^0$ and $g^3$,

$$\begin{aligned} \mu\frac{dg}{d\mu} &= \frac{\rho^3}{16\pi^2}\left(\frac{11}{6}n - \frac{1}{3}n_f\right)(1-3) = \\ &= \boxed{-\frac{g^3}{16\pi^2}\left(\frac{11}{3}n - \frac{2}{3}n_f\right)} \\ &=: \beta(g) \end{aligned} \tag{21.11}$$

which is negative for small $n_f$. This result is due to Politzer, Gross and Wilczek, who received the 2004 Nobel prize for this achievement, and Szymanzik and 't Hooft.

## QCD

In QCD, $n = 3$ and $n_f = 6$:

$$\boxed{\frac{g^2(\mu^2)}{4\pi} = \frac{g^2(\mu_0^2)/4\pi}{1 + \frac{g^2(\mu_0^2)}{4\pi}\frac{1}{12\pi}(33 - 2n_f)\log(\mu^2/\mu_0^2)}} \tag{21.12}$$

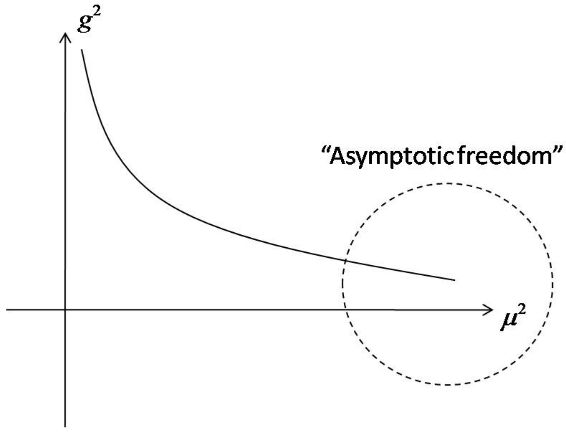



Figure 21.1: Asymptotic freedom

**Note**

$$\Lambda = \mu \exp\left[-\int_{g(\mu_0^2)}^{g(\mu^2)} \frac{dg}{\beta(g)}\right]$$

does *not* depend on $\mu$. The transition from $g(\mu_0)$ to $\Lambda$ is called *dimensional transmutation.*

In the absence of a quark mass, there is no scale in the theory, so one is free to choose an arbitrary scale: $g$ is not fixed, but fixing $g(\mu_0^2)$ gives a scale $\mu_0^2$, or $\Lambda$. (Note: $g^2(m_\rho^2)$, where $m_\rho^2$ is a QCD bound state mass squared, can in principle be calculated!) This can only be realized in the present framework of perturbation theory for small $g(\mu^2)$, i.e. for large $\mu^2$, or small distances. Otherwise, one needs *non-perturbative* methods ("IR slavery"), like lattice gauge theory. Since $\mu$ appears in $(g^2/4\pi)\log(p^2/\mu^2)$, it corresponds to the scale of the problem.

**Note**

$p^2$ and $\mu^2$ should not differ too much, otherwise the logarithm is big - and in spite of the small magnitude of $\alpha = g^2/4\pi$, one has to consider all orders of $(g^2/4\pi)\log(...)$ (this is called the 'leading log approximation'; see fig. 21.2).

We have to take the scale of the problem into account. In a manner of speaking, the problem chooses its own scale, i.e. the scale cannot be freely chosen. We will adress this when discussing the *renormalization group.* Before doing this, however, it is advisable to do some calculations in QED, which we will do in the next chapter.

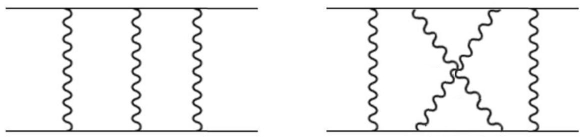


Figure 21.2: Leading log approximation (left) and $\alpha$-suppressed contribution (right)

Whereas in QCD the asymptotic freedom behavior justifies the use of perturbation methods at small distances, this is not justified for QED in the same way: there, the coupling is small in the IR and grows slowly in the UV. Later we will see the abstraction of the QCD lattice action fromt he continuum classical action.

Now, before we go to the renormalization group, let us practise a bit with QED.

# Chapter 22

# One-loop QED

Let us explicitly perform some calculations from one-loop QED. This is both historically and practically important: historically, since this is the 'mother of quantum field theory', and practically, to get some experience with actual calculations in QFT. This subject will be treated extensively in the exercises as well.

This chapter is based on Peskin & Schöder, Itzykson & Zuber, and Ramond's treatments of the topic; the images come from Peskin & Schröder.

For QED, one can use the Feynman rules for non-abelian gauge theories with trivial structure coefficients.

## 22.1 Self-energy

Let us begin by calculating the self-energy, to first loop order, of the electron propagator:

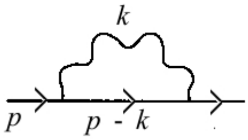



Figure 22.1: First order correction to electron propagator

We will use the Feynman gauge, $\xi = 1$, and have $\lambda_a/2 = 1$. Then,

$$\Sigma(p) = (ig)^2 i \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu(\not{p} - \not{k} + m)\gamma_\nu}{((p-k)^2 - m^2 + i\epsilon)(k^2 + i\epsilon)} \tag{22.1}$$

Going to Euclidean spacetime,

$$\Sigma(p) = -g^2 \int \frac{d^4k_{\mathrm{E}}}{(2\pi)^4} \frac{\gamma^\nu(-(\not{p} - \not{k}) + m)\gamma_\nu}{((p-k)^2 + m^2)k^2}$$

Applying dimensional regularization, and using some of the $\gamma$-algebra relations ($\gamma_\mu\gamma_\mu = -2\omega$; $\gamma_\mu\gamma_\rho\gamma_\mu = (2\omega-2)\gamma_\rho$),

$$\begin{aligned}
\Sigma(p) &= -g^2(\mu^2)^{2-\omega}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{-2\omega m - 2(\omega-1)(\not{p}-\not{k})}{((p-k)^2+m^2)k^2} = \\
&= -g^2(\mu^2)^{2-\omega}\int_0^1 dx\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{-2\omega m - 2(\omega-1)(\not{p}(1-x)-\not{k}')}{((k')^2+p^2x(1-x)+m^2x)^2} \\
&= 2g^2(\mu^2)^\epsilon\int_0^1 dx\,\{\not{p}(1-x)+2m-\epsilon(\not{p}(1-x)+m)\}\times \\
&\qquad \frac{\pi^\omega}{(2\pi)^{2\omega}}\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{1}{(p^2x(1-x)+m^2x)^\epsilon}
\end{aligned}$$

where in the second step, we have used $k' = k - xp$, and in the third, $\epsilon = 2-\omega$. Sending $\epsilon \to 0$, this becomes

$$\begin{aligned}
&\frac{2g^2}{16\pi^2}\Gamma(\epsilon)\int_0^1 dx\,\{\not{p}(1-x)+2m-\epsilon(\not{p}(1-x)+m)\}\times \\
&\quad \exp\left\{-\epsilon\log\left(\frac{p^2x(1-x)+m^2x}{\mu^2}\right)\right\}(1-\epsilon\log 4\pi) = \\
&\frac{g^2}{16\pi^2}\frac{\not{p}+4m}{\epsilon} - \frac{g^2}{8\pi^2}\left[\left(\frac{1}{2}\not{p}(1+\gamma+\log 4\pi)+m(1+2\gamma+2\log 4\pi)\right)+\right. \\
&\qquad \left.\int_0^1 dx\left\{(\not{p}(1-x)+2m)\log\frac{p^2x(1-x)+m^2x}{\mu^2}\right\}\right]
\end{aligned}$$

**Counterterms**

The counterterm coming from the electron wavefunction renormalization, in Euclidean spacetime, is given by

$$-i\cdot i(Z_2-1)(\not{p}+m)$$

where the factor of $i$ comes from the Euclidean transformation $k_0 \to ik_4$. From the mass renormalization, we have

$$-i\cdot i\delta m$$

Let us apply minimal subtraction (MIN), including the finite terms that appear in every factor (MIN'). We obtain:

$$\begin{aligned}
Z_2 - 1 &= -\frac{g^2}{16\pi^2\epsilon} + \text{finite terms} \\
\delta m &= -\frac{3mg^2}{16\pi^2\epsilon} + \text{finite terms}
\end{aligned}$$

The renormalization conditions (on mass shell, "$\not p = m$") are:

$$\Sigma(\not p)|_{\not p = m} = 0 \qquad \text{and}$$
$$\frac{\partial}{\partial \not p}\Sigma(\not p)\Big|_{\not p = m} = 0$$

In Itzykson & Zuber, this is represented as

$$\Sigma(\not p) = A(p^2) + \not p B(p^2), \qquad \text{with} \qquad A(p^2 = m^2) = B(p^2 = m^2) = 0$$

## 22.2 Vertex

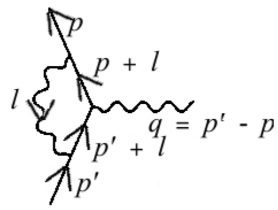



Figure 22.2: First order correction to the QED vertex

In Euclidean spacetime, this graph has the expression

$$\begin{aligned}
\Gamma_\rho =& i^6 i(-1)^3 (g\mu^{2-\omega})^3 \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \gamma_\tau \frac{1}{\not p' + l + m} \gamma_\rho \frac{1}{\not p + l + m} \gamma_\sigma \frac{\delta_{\sigma\tau}}{l^2} = \\
& - 2i(g\mu^{2-\omega})^3 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega}}{(2\pi)^{2\omega}} \times \\
& \qquad \frac{\gamma_\tau (\not p' + \not l - m)\gamma_\rho (\not p + \not l - m)\gamma_\tau}{(l^2 + m^2(x+y) + 2l(p'x + py) + p'^2 x + p^2 y)^3} = \\
& - 2i(g\mu^{2-\omega})^3 \int_0^1 dx \int_0^{1-x} dy \left(\Gamma^{(2)} + \Gamma^{(1)}\right)
\end{aligned} \tag{22.2}$$

with

$$\Gamma^{(2)} = \frac{\Gamma(3-\omega)}{(4\pi)^\omega \Gamma(3)} \frac{\gamma_\tau(\not p'(1-x) - \not p y - m)\gamma_\rho(\not p(1-y) - \not p' x - m)\gamma_\tau}{(m^2(x+y) + p'^2 x + p^2 y - (p'x + py)^2)^{3-\omega}}$$
$$\Gamma^{(1)} = \frac{1}{2}\frac{\Gamma(2-\omega)}{(4\pi)^\omega \Gamma(3)} \frac{\gamma_\tau \gamma_\nu \gamma_\rho \gamma_\nu \gamma_\tau}{(m^2(x+y) + p'^2 x + p^2 y - (p'x + py)^2)^{2-\omega}}$$

In the last step in eq. 22.2, we have performed a shift in the integration variable:

$$l' = l + p'x + py \qquad \text{so}$$
$$\not l \rightarrow -\gamma(p'x + py)$$

Now, we use

$$\begin{aligned}\gamma_\tau\gamma_\alpha\gamma_\rho\gamma_\beta\gamma_\tau =& 2\gamma_\beta\gamma_\rho\gamma_\alpha - 2\underbrace{(2-\omega)}_{\epsilon}\gamma_\alpha\gamma_\rho\gamma_\beta \\ \gamma_\nu\gamma_\rho\gamma_\nu =& -2(1-\omega)\gamma_\rho\end{aligned}$$

Then, $\Gamma^{(2)}$ is finite for $\omega \to 2$. The only divergent contribution to $\Gamma_\rho$ then comes from a term in $\Gamma^{(1)}$:

$$-2ig^3\left(1+\epsilon\frac{3}{2}\log\mu^2\right)\frac{1}{2}\left(\frac{1}{\epsilon}-\gamma\right)\frac{1+\epsilon\log 4\pi}{2(4\pi)^2}\times \\ \frac{\gamma_\tau\gamma_\nu\gamma_\rho\gamma_\nu\gamma_\tau}{(m^2(x+y)+p'^2x+p^2y-(p'x+py)^2)^\epsilon}$$

If one is only interested in the terms containing $\epsilon^{-1}$ and $\log\mu^2$, one sends $\omega \to 2$ in the last fraction.

Now, let us consider $\Gamma^{(2)}$ "on shell", i.e. sandwiched between $\bar{u}(p')$ and $u(p)$. Consider

$$\begin{aligned}\gamma_\rho \not{p} =& -p_\rho - 2i\sigma_{\rho\tau}p_\tau = -m\gamma_\rho + \gamma_\rho(\not{p}+m) \\ \not{p}'\gamma_\rho =& -p'_\rho + 2i\sigma_{\rho\tau}p'_\tau = -m\gamma_\rho + (\not{p}'+m)\gamma_\rho\end{aligned}$$

(Remember, $\sigma_{\rho\tau} = \frac{-i}{4}[\gamma_\rho,\gamma_\tau]$ and $\{\gamma_\rho,\gamma_\tau\} = 2\delta_{\rho\tau}$.) $\not{p}+m$ and $\not{p}'+m$ give zero when acting on $u(p)$ and $\bar{u}(p')$, respectively. This is the so-called *Gordon identity*:

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)_\mu}{2m} + i\frac{\sigma^{\mu\nu}q^\nu}{2m}\right]u(p) \tag{22.3}$$

The part of $\Gamma^{(2)}$ containing $\sigma_{\mu\nu}$ then becomes

$$\Gamma^{(2)}_{(\sigma_{\mu\nu})} = 8im\left[\sigma_{\rho\tau}p_\tau(-x+y(y+x)) - \sigma_{\rho\tau}p'_\tau(-y+x(y+x))\right]$$

and the contribution to $\Gamma_\rho$ for $p_\mathrm{E}^2 = {p'_\mathrm{E}}^2 = -m^2$ ($p_\mathrm{E}p'_\mathrm{E} = -m^2$) and $q^2 = 0$

$$\Gamma_\rho^{(\sigma_{\mu\nu})} = -i\frac{g^3}{(4\pi)^2}\int_0^1 dx\int_0^{1-x} dy\, 8\,im\times \\ \frac{\sigma_{\rho\tau}p_\tau(-x+y(y+x)) - \sigma_{\rho\tau}p'_\tau(-y+x(y+x))}{m^2(x+y)^2}$$

Investigating the first part of the integrand, we see that

$$\int_0^1 dx\int_0^{1-x} dy\frac{x-y(x+y)}{(x+y)^2} = \int_0^1 dx\int_x^1 d(x+y)\frac{x-((x+y)-x)}{(x+y)^2} = \\ -\int_0^1 dx(\log x)x = \frac{1}{4}$$

A similar identity holds for the second part, yielding the final result

$$\Gamma_\rho^{\sigma_{\mu\nu}} = \frac{g^3}{8\pi^2 m}\sigma_{\rho\tau}(p'-p)_\tau$$

This is the contribution to the magnetic moment. In QED, $g$ is set to be $e$, and this becomes

$$\frac{e}{m}\left(\frac{\alpha}{2\pi}\right)\sigma_{\rho\tau}(p'-p)_\tau$$

(with $\alpha = e^2/4\pi$ the fine structure constant). Then,

$$\frac{e}{m}\sigma_{\rho\tau}(p'-p)_\tau A_\rho = \frac{ie}{2m}\sigma_{\rho\tau}F_{\rho\tau}$$

gives the tree-level (i.e. $g = 2$) contribution to the intrinsic magnetic moment in units of the Bohr magneton times spin $\frac{1}{2}$, $\frac{e}{2m}\frac{1}{2}$. The one-loop correction is given by

$$\boxed{g = 2\left(1+\frac{\alpha}{2\pi}\right)} \qquad (22.4)$$

This is the equation that Feynman, Schwinger and Tomonaga got the Nobel prize for. Nowadays, many-loop contributions have been calculated.

## 22.3 $\gamma_\rho$ contribution, infrared singularity

Having resolved the $\sigma_{\rho\tau}$-contribution, we still have the $\gamma_\rho$-part sandwiched between $\bar{u}$ and $u$:

$$\Gamma_\rho^{(1)} - ig\gamma_\rho\frac{g^2}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy \times \\ \frac{2m^2((x+y)^2 - 2(1-x-y) - 2q^2(1-x)(1-y))}{[m^2(x+y)+p'^2x(1-x)+p^2y(1-y)-2xypp']^{3-\omega}}$$

$\Gamma_\rho^{(1)}$ has to be renormalized, but does not have any singularities in the $x$- and $y$-integrals. Evaluating the second term at $q^2 = 0$, or $pp' = -m^2$, we see that the integral

$$\int_0^1 dx\int_0^{1-x} dy\frac{2m^2((x+y)^2-2(1-x-y))}{[m^2(x+y)-m^2(x(1-x)+y(1-y))+2xym^2]^{3-\omega}} = \\ \int_0^1 dx\int_0^{1-x} dy\frac{2m^2((x+y)^2-2(1-x-y))}{[m^2(x^2+y^2+2xy)]^{3-\omega}}$$

Where $x$ and $y$ are close to zero, this is divergent:

$$\int_0^1 dx\int_0^{1-x} dy\frac{-4}{((x+y)^2)^{3-\omega}} = -4\left[\frac{1}{2\omega-5}\left(1-\frac{1}{(2\omega-4)}\right)\right] \approx -4\left(-\frac{1}{\epsilon}\right)$$

However, this is an IR singularity: it does not arise from our ignorance about the physics at very large momenta, but rather at very low momenta. Therefore, it should not be dealt with like the UV singularities we have been treating so far (expansion in $\mu^\epsilon$, $(4\pi)^\omega$, etc.). Instead, we will introduce a "photon mass" $\lambda$. The point is that these singularities have physical origins: in QED, one cannot distinguish a lone electron from an electron accompanied by a photon of arbitrary low energy (also called "soft photon"). In QCD, the same problem arises, and furthermore, it is impossible to tell apart two almost collinear gluons from one.

So, one has to take into account the measurement process when calculating cross sections (for example for jets in QCD): "soft" photons are not observed, and have to be added to the process' virtual photoproduction. Adding up these contributions, we obtain finite cross sections (see exercise).

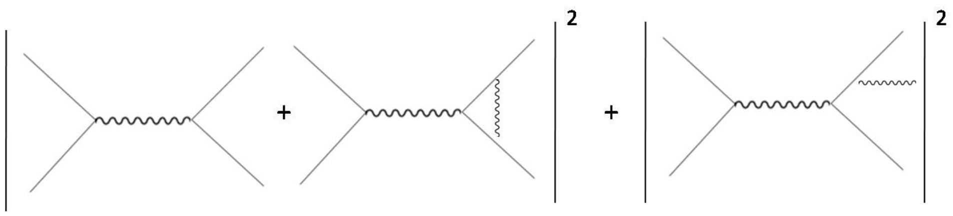


Figure 22.3: Modification of cross section by soft photon emission

If $q^2 \neq 0$, the $\gamma_\rho$ part produces a "charge form factor" $F_1(q^2)$. Everything taken together gives

$$e\bar{u}(p)\Gamma^\mu(p',p)u(p) = e\bar{u}(p')\left(\gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right)u(p)$$

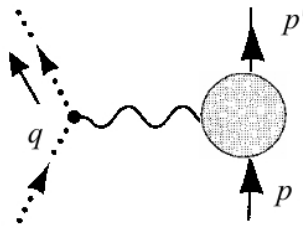
Given the renormalization condition, we can require that $F_1(q^2 \to 0) = 1$.

## 22.4 Ward-Takahashi identity

Reducing an outer (on shell) photon with polarization $\epsilon_\mu$ in the LSZ-formalism - very similar to the bosonic case we did in some detail - we obtain, apart from the other outer particle fields,

$$\langle k_\gamma f|i\rangle \to \epsilon^\mu \int d^4x e^{-ik_\gamma x}(-\partial_x^2)\langle 0|\,\mathrm{T}(\dots A_\mu(x)\dots)\,|0\rangle = \epsilon^\mu \mathbf{M}_\mu$$

Gauge invariance of the amplitude requires invariance under $\epsilon^\mu \to \epsilon^\mu + \alpha k^\mu$, i.e. $k^\mu \mathbf{M}_\mu = 0$. This can also be shown to be true if $k^\mu$ is off-shell, like for $q^\mu$ before.




Now recall the Ward-Takahashi identity for symmtries of a scalar field, from chapter 18. This symmetry was also related to electric charge ($\varphi \to e^{i\alpha}\varphi$). Here, with two outer fermions (electrons), it reads:

$$\begin{aligned}-\partial_{\mu_x} \langle 0|\, \mathrm{T}(j^\mu_{\mathrm{sy}}(x)\Psi_\alpha(y)\bar{\Psi}_\beta(z))\,|0\rangle &= \\ i\delta(x-y)\,\langle 0|\, \mathrm{T}(\Psi_\alpha(y)\bar{\Psi}_\beta(z))\,|0\rangle &- \\ i\delta(x-z)\,\langle 0|\, \mathrm{T}(\Psi_\alpha(y)\bar{\Psi}_\beta(z))\,|0\rangle &\end{aligned} \tag{22.5}$$

with the Noether symmetry current

$$j^\mu_{\mathrm{sy}} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi_\alpha)}\delta\Psi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi}_\beta)}\delta\bar{\Psi}_\beta$$

and

$$\delta\Psi_\alpha = -ie\Psi_\alpha, \qquad \delta\bar{\Psi}_\beta = ie\bar{\Psi}_\beta$$

Now inspecting the $Z$-factors in the Lagrangian from chapter 21 for the abelian case, we obtain

$$j^\mu_{\mathrm{sy}} = -ieZ_2\bar{\Psi}(i\gamma^\mu)\Psi = eZ_2\bar{\Psi}\gamma^\mu\Psi$$

(with renormalized fields). In the three-point function

$$\langle 0|\, \mathrm{T}(j^\mu(x)\Psi_\alpha(y)\bar{\Psi}_\beta(z)\,|0\rangle\,,$$

$j^\mu$ is shorthand notation for $-\partial^2 A^\mu = eZ_{1\mathrm{F}}\bar{\Psi}\gamma^\mu\Psi$, according to the equation of motion. The renormalization of the gauge field itself does not play a role in this consideration where just its coupling is important: $j^\mu_{\mathrm{sy}} = j^\mu Z_2/Z_{1\mathrm{F}}$.

Fourier transforming the above gives

$$q_\mu\tilde{S}(p')\frac{\Gamma^\mu}{i}(p',p)\tilde{S}(p) = \frac{Z_{1\mathrm{F}}}{Z_2}e[\tilde{S}(p) - \tilde{S}(p')]$$

or the *Ward-Takahashi identity*:

$$\boxed{q_\mu\frac{\Gamma^\mu}{i}(p',p) = \frac{Z_{1\mathrm{F}}}{Z_2}e[\tilde{S}^{-1}(p') - \tilde{S}^{-1}(p)]} \tag{22.6}$$

Since $\Gamma^\mu$ and $\tilde{S}$ are all finite quantities after renormalization,

$$\boxed{Z_{1\mathrm{F}} = Z_2}$$

in minimal subtraction.

Sandwiching eq. (22.6) between $\bar{u}(p')$ and $u(p)$ and using the Dirac equation for $\bar{u}$ and $u$, the right-hand side gives zero, and we find

$$eq^\mu \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p) = 0$$

which is easy to check, since $\sigma^{\mu\nu}$ is antisymmetric, and for the $\gamma^\mu$-part, we can use the Dirac equation. $F_1(q^2 = 0) = 1$ is the usual normalization condition, defining the coupling constant to be $g = e$.

### Remarks

- More extensive treatments of the Ward-Takahashi idendity can be found in textbooks (e.g. Peskin & Schröder).
- The anomalous magnetic moment of the electron has been tested to extremely high precision ($\alpha^{-1}$ is known up to 10 significant digits).
- Another important experimental test for QED is the Lamb shift effect in the hydrogen atom.

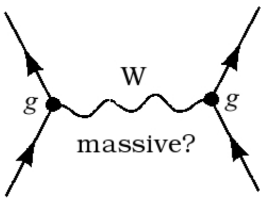
$$\begin{array}{rl} 1057 & \text{experimental result} \\ \left.\begin{array}{r} 1010 \\ 68 \\ -27 \end{array}\right. & \left.\begin{array}{l} \text{form factor vertex correction} \\ \text{anomalous moment} \\ \text{photon propagator} \end{array}\right\} \text{theoretical prediction} \\ 1052 & \text{total, excluding } O(\alpha^2) \text{ correction} \end{array}$$

  In this case, the perturbative expansion is not around the free electron field, but around the hydrogen state.

# Chapter 23

# Spontaneous symmetry breaking and the Higgs mechanism

In gauge theories, the gauge bosons are massless even after renormalization. Now, weak interactions are obviously not mediated by massless fields. Nonetheless, we would like to be able to describe these interactions by gauge theories. The beauty of such theories, however, is precisely the rigid structure of couplings fixed by the symmetries of the theory, which we would like to keep intact. The solution to this problem is *spontaneous symmetry breaking*: a spontaneously broken gauge symmetry preserves this structure.




## 23.1 Higgs mechanism for abelian symmetry ($U(1)$)

Let us begin with the Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |\underbrace{(\partial_\mu - ieA_\mu)}_{\mathcal{D}_\mu}\Phi|^2 - V(\Phi) \qquad \text{with} \tag{23.1}$$

$$V(\Phi) = -\mu^2\Phi^*\Phi + \frac{\lambda}{2}(\Phi^*\Phi)^2 \qquad \left(= \frac{\lambda}{2}\left(\Phi^*\Phi - \frac{\mu^2}{\lambda}\right)^2 - \frac{\mu^4}{2\lambda}\right)$$

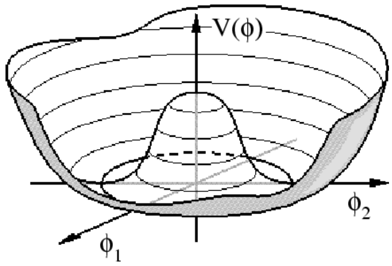



Figure 23.1: Mexican hat potential

This is called the *Higgs Langrangian*, which is frequently used in elementary particle physics. It is also known as the *Ginzburg-Landau Lagrangian*: it occurs in the theory of superconductivity, with non-relativistic $\Phi$. It has the famous Mexican hat potential:
It has a minimum at $\langle\Phi\rangle = \Phi_0 = \sqrt{\mu^2/\lambda}$, which is fixed up to a phase $e^{i\chi}$. This is where the spontaneous breaking of the $U(1)$-symmetry occurs.

**Mass terms**

Expanding $\Phi$ around its minimum,

$$\Phi(x) = \Phi_0 + \frac{1}{\sqrt{2}}[\Phi_1(x) + i\Phi_2(x)],$$

we can write the potential as

$$V(\Phi) = -\frac{\mu^4}{2\lambda} + \mu^2\Phi_1^2 + \mathcal{O}(\Phi_i^3)$$

Here, we see a boson ($\Phi_1$) with mass $2\mu^2$, and a massless Goldstone boson ($\Phi_2$). The term

$$e^2 A_\mu A^\mu \Phi^*\Phi \to e^2 A_\mu A^\mu \Phi_0^2 \quad \text{with} \quad \Phi_0^2 = \frac{\mu^2}{\lambda}$$

gives a mass $m_A^2 = 2e^2\mu^2/\lambda$ to the gauge bosons. The cross term

$$2i\frac{k_\mu}{i} e A_\mu \frac{\Phi_0}{\sqrt{2}}\Phi_2 = m_A k_\mu A^\mu \Phi_2$$

gives a contribution to the gauge boson self-energy (vacuum polarization):

 $= m_A k_\mu \frac{i}{k^2}(-m_A k_\mu)$

 $= i m_A^2 g_{\mu\nu}$

These two diagrams add up to

$$\underbrace{im_A^2\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}_{\Pi_{\mu\nu}}$$

One could say that the Goldstone boson $\Phi_2$ is "eaten" by the gauge boson to produce a massive gauge boson. In other words, $\Phi_2$ is transformed away by a local gauge transformation. This mechanism is also at work in superconductivity, where the photon acquires an effective mass; this is equivalent to having a finite penetration depth for the magnetic field.

## 23.2 Higgs mechanism for $SU(2)$ gauge symmetry

In the case of $SU(2)$ gauge symmetry, the field $\Phi$ is a doublet $(\Phi_1\ \Phi_2)^T$ of complex scalar (Higgs) fields, and the covariant derivative and potential are given by

$$\begin{aligned}
\mathcal{D}_\mu\Phi &= \left(\partial_\mu \mathbb{1}_2 - ig\frac{\tau^a}{2}A_\mu^a(x)\right)\Phi \qquad \text{and} && (23.2)\\
V(\Phi) &= -\mu^2\Phi^\dagger\Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2 \qquad \text{with} && (23.3)\\
\Phi_0^\dagger\Phi_0 &= \frac{\mu^2}{\lambda} \qquad \text{at the minimum} && (23.4)
\end{aligned}$$

Choosing the expectation value of $\Phi$ to be real, we obtain

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \text{with} \qquad \frac{v^2}{2} = \frac{\mu^2}{\lambda}.$$

Rewriting this in the mass-squared form seen above,

$$\begin{aligned}
\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & v \end{pmatrix}\tau^b A_\mu e^b \frac{g^2}{4}\tau^a A_\mu e^a \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} &= \frac{g^2}{8}A_\mu e^a A_\mu e^a v^2 \qquad \text{so}\\
m_A^2 &= \frac{1}{2}g^2\frac{\mu^2}{\lambda}
\end{aligned}$$

This is the mass of the W-boson.

Now, consider the Higgs potential:

$$\begin{aligned}
\Phi^\dagger\Phi &= \frac{(\Phi^i)^2}{2} \qquad \text{with}\\
\Phi^{1,3} &= \Re\left(\frac{\Phi_{1,2}}{\sqrt{2}}\right), \qquad \Phi^{2,4} = \Im\left(\frac{\Phi_{1,2}}{\sqrt{2}}\right)
\end{aligned}$$

The minimum then has $\Phi_0^v = (0,\ 0,\ 0,\ v)$ after spontaneously breaking the $SU(2)$-symmetry. In general,

$$\Phi^i = (\pi^1,\ \pi^2,\ \pi^3,\ v + \sigma = \pi^4),$$

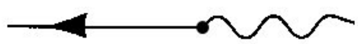
where the $\pi^i$ are the massless Goldstone bosons.

The mass terms in the Lagrangian are modified by

$$\mathcal{L} = \cdots - \frac{1}{2}(2\mu^2)\sigma^2$$

so there are three Goldstone bosons and a massive Higgs particle $\sigma$ with $m_\sigma^2 = 2\mu^2$.

Inspecting the mixing diagram



we see

$$\begin{aligned} -gk^\mu (A^a)^\mu \Re(\Phi \tau^a \Phi_0^*) &= -gk^\mu (A^a)^\mu \Re \left[ \begin{pmatrix} \frac{\pi_1 + i\pi_2}{\sqrt{2}} & \frac{\pi_3}{\sqrt{2}} \end{pmatrix} \tau^a \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \right] = \\ gk^\mu (A^i)^\mu \pi_i \frac{v}{2} &= m_A k^\mu (A^i)^\mu \pi_i \end{aligned}$$

Again, the selfenergy is given by

$$\left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) m_A^2 + \mathcal{O}(k^2)$$

We get three gauge bosons $A_\mu^\pm = (A_\mu^1 \pm i A_\mu^2)/\sqrt{2}$ and $A_\mu^3$, which couple to $\pi^\pm = (\pi_1 \pm i\pi_2)/\sqrt{2}$ and $\pi_3$.

## 23.3 Electroweak theory

The electroweak theory is a $SU(2)_\mathrm{W} \times SU(1)_\mathrm{Y}$ gauge theory with a doublet Higgs-boson.

$$\mathcal{D}_\mu \Phi = \left( \partial_\mu \mathbb{1}_2 - ig\frac{\tau^a}{2} A_\mu^a(x) - \frac{i}{2} g' B_\mu(x) \right) \Phi$$

with $g'/2 = g' Y_\Phi$, where $Y_\Phi$ is the hypercharge of the $\Phi$-field. The W-bosons $A_\mu^\pm$ and a mixture of $A_\mu^3$ and $B_\mu$ become massive: the latter gives the Z-boson. Other combinations give the electromagnetic field, which is massless. The $SU(2)_\mathrm{W}$ gauge bosons couple to the left-handed quarks and leptons:

$$\begin{aligned} \bar{\Psi} i \partial\!\!\!/ \Psi &= \bar{\Psi}_L i \partial\!\!\!/ \Psi_L + \bar{\Psi}_R i \partial\!\!\!/ \Psi_R \qquad \text{with} \\ \Psi_L &= \left( \frac{1 \pm \gamma_5}{2} \right) \Psi \end{aligned}$$

Now, substitute $\partial\!\!\!/ \to D\!\!\!\!/$ in the $\Psi_L$-part, where the covariant derivative acts in the representation of quarks and leptons (again a doublet for $SU(2)$, with a different hypercharge quantum number $Y$). Then,

$$\boxed{Q_{\text{el}} = T_3 + \frac{Y}{2}}$$

with $T_3$ an $SU(2)_W$ generator. The coupling of the quarks and leptons to the Higgs-bosons provides quark/lepton mass:

$$f\bar{\Psi}_L\Psi_R\langle\Phi\rangle$$

where the contraction gives an $SU(2)_W$ singlet.

## 23.4 Perturbation theory in electroweak theory

When doing perturbation theory in electroweak theory, one includes $\langle\Phi\rangle$ in the Feynman rules. Be careful *not* to eliminate the Goldstone bosons, as was done earlier, in the *unitary gauge*. A very efficient gauge to work in is the 't Hooft (background) gauge, which introduces a term

$$\frac{1}{2\xi}\left(\bar{D}^\mu A^a_\mu - \xi g\frac{v}{2}\pi^a\right)^2$$

$\bar{D}$ contains a background field $\bar{A}_{\text{bg}}$, which is defined by

$$A^a_\mu = \bar{A}^a_\mu + a^a_\mu$$

where $a^a_\mu$ is a quantum field, which gives the deviation of $A^a_\mu$ from the background $\bar{A}^a_\mu$.

A local gauge transformation

$$\delta A^a_\mu = \partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c$$

can be imagined to act on $\bar{A}^a_\mu$ or $a^a_\mu$ inhomogeneously. The other part can be transformed homogeneously (by a tensor type transformation). If it acts on $a^a_\mu$, this can be gauge fixed by a covariant background gauge $\bar{D}^\mu a^a_\mu - \omega^a = 0$ (or with $v\pi^a$ instead of $\omega^a$ for the electroweak interaction). The resulting Lagrangian with gauge fixing term is invariant under a gauge transformation which acts as such on $\bar{A}^a_\mu$ and as a homogeneous transformation on $a^a_\mu$ ($a^a_\mu$ is a "matter field" in this context).

# Chapter 24

# Renormalization group, Wilson renormalization

As an introduction to the topic, let us look at *Weinberg's theorem* for Feynman diagrams in renormalizable theories. It says that $n$-point functions in the "deep Euclidean" (where $p_i^2 = -c_i p^2$ and $p^2 \to \infty$) scale with

$$\boxed{(p^2)^{\omega/2} P(\log(p^2/\mu^2))}$$

where $\omega$ is the superficial degree of divergence and $P$ is some power series. (Note that in this limit, mass terms are irrelevant.)

The problem that arises here is whether the logarithmic terms from perturbation theory add up to powers.

## 24.1 Callan-Szymanzik equation

For 1PI, truncated, renormalized $n$-point functions, we have the following relation:

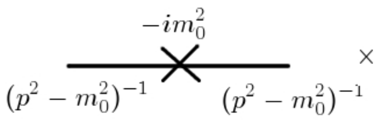
$$\Gamma^{(n)}(p_1, \dots, p_n) = Z^{n/2} \Gamma_{\mathrm{u}}^{(n)}(p_1, \dots, p_n)$$

(The subscript u stands for 'unrenormalized', and truncation means taking out the outer propagators.) This relation reverses the sign of the power of $\sqrt{Z}$: earlier, we had $\Phi_{\text{ren.}} = Z^{-1/2}\Phi_{\mathrm{u}}$.

Inserting a mass term $\Delta = m_0^2 \Phi_0^2/2$ into the unrenormalized Lagrangian and defining

$$i\Gamma_{\mathrm{u}\Delta}^{(n)}(p_1, \dots, p_n) = m_0 \frac{\partial}{\partial m_0} \Gamma_{\mathrm{u}}^{(n)}(p_1, \dots, p_n)$$

this corresponds to inserting a mass vertex in the propagators of an unrenormalized perturbative $\Gamma$ in all possible ways, since




$$\times\, i = \frac{m_0}{2} \partial_{m_0} (p^2 - m_0^2)^{-1}$$

This is an argument from perturbation theory. However, we do not want to restrict ourselves to finite order in perturbation theory here. We define

$$\Gamma^{(n)}_{\Delta}(p_1, \ldots, p_n) := c Z^{n/2} \Gamma^{(n)}_{\mathrm{u}\Delta}(p_1, \ldots, p_n)$$

with a cut-off (regulator) dependent $c$ such that the left hand side is cut-off-independent. This yields the *Callan-Szymanzik equation*:

$$\boxed{i\Gamma^{(n)}_{\Delta}(p_1, \ldots, p_n)} = c Z^{n/2} \frac{m_0}{2} \frac{\partial}{\partial m_0} Z^{-n/2} \Gamma^{(n)}(p_1, \ldots, p_n) = \tag{24.1}$$

$$\boxed{\left[ m \frac{\partial}{\partial m} + \beta(\lambda) \frac{\partial}{\partial \lambda} - n\gamma(\lambda) \right] \Gamma^{(n)}(p_1, \ldots, p_n)} \tag{24.2}$$

with $c$ defined such that

$$c \frac{m_0}{2} \frac{\partial m}{\partial m_0} = m$$

and, if $\Gamma^{(n)}$ depends on $m$ and $\lambda$,

$$\beta = c \frac{m_0}{2} \frac{\partial \lambda}{\partial m_0} \tag{24.3}$$

$$\gamma = \frac{c}{2} \frac{m_0}{2} \frac{\partial \log Z}{\partial m_0} \tag{24.4}$$

Dimensional analysis shows that

$$\Gamma^{(n)}(p_1, \ldots, p_n) = f\left( \frac{p_i p_j}{p^2}, \lambda, \frac{m^2}{p^2} \right) (p^2)^{(4-n)/2}$$

$$\Gamma^{(n)}_{\Delta}(p_1, \ldots, p_n) \propto (p^2)^{-1+(4-n)/2}$$

where the $-1$ term in the exponent of $p^2$ in $\Gamma^{(n)}_{\Delta}$ comes about due to the presence of an extra factor of $m_0^2/p^2$. This implies that for large $p^2$, we can neglect $\Gamma^{(n)}_{\Delta}(p_1, \ldots, p_n)$.

## 24.2 Renormalization group equations

In the above, we have taken the physical mass $m$ as renormalization point. In general, and in particular for massless theories, we would like to introduce a renormalization scale $\mu$ and obtain a fixed $m$ in terms of $\mu$ and $\lambda$. In this case, we have to substitute $\mu\partial_\mu$ for $m\partial_m$ in the Callan-Szymanzik equation, i.e. fix $c$ by $c\frac{m_0}{2}\partial_{\mu_0}\mu = \mu$. The left hand side $\Gamma_\Delta^{(n)}$ can be neglected for large $p^2$, or, equivalently, small $m^2$, which includes massless theories, in the original CS equation.

This also happens for the so-called renormalization group equation, whose derivation is very similar to the QCD case discussed in chapter 21. To see this, let us go back to our well-beloved $\Phi^4$ theory, where we have

$$\begin{aligned} \Gamma_{\mathrm{u}}^{(n)} &= (Z_2)^{-n/2}\Gamma^{(n)} \quad \text{with} \\ Z_2 &= Z_2(\lambda, m/\mu) \quad \text{and} \\ \lambda_0 &= \lambda\frac{Z_1}{Z_2^2} \end{aligned}$$

with physical mass $m$ and renormalization point $\mu$. $\lambda$ and $m$ are renormalized quantities, whereas $\lambda_0$ and $m_0$ are the bare quantities, which in the renormalized theory depend on $\lambda$, $m$, $\mu$ and the regulator ($\epsilon$, $\Lambda$, Pauli-Villars).

At fixed $\lambda_0$ and $m_0$, physical results should not depend on $\mu$, since it does not appear in $\Gamma_{\mathrm{u}}^{(n)}$. That is,

$$0 = 2\mu^2\frac{\partial}{\partial\mu^2}\Gamma_{\mathrm{u}}^{(n)} = -\frac{n}{2}Z_2^{-n/2}2\mu^2\left(\frac{\partial}{\partial\mu^2}\log Z_2\right)\Big|_{\lambda_0,m_0}\Gamma^{(n)} + \underbrace{Z_2^{-n/2}2\mu^2\,\frac{\partial}{\partial\mu^2}\Gamma^{(n)}\Big|_{\lambda_0,m_0}}_{Z_2^{-n/2}\left[2\mu^2\,\frac{\partial}{\partial\mu^2}\Gamma^{(n)}\big|_{\lambda,m^2}+2\mu^2\,\frac{\partial\lambda}{\partial\mu^2}\big|_{\lambda_0,m_0}\,\frac{\partial}{\partial\lambda}\Gamma^{(n)}\big|_{\mu^2,m^2}\right]}$$

where $\lambda_0$, $m_0$ and $\Gamma^{(n)}$ are all functions of $\lambda$, $m^2$ and $\mu^2$, and

$$\frac{\partial m^2}{\partial\mu^2}\Big|_{\lambda_0,m_0} = 0$$

In other words, the physical pole in the propagator is fixed at fixed values of $\lambda_0$, $m_0$ and the regulator. From the above, we have that

$$\left\{2\mu^2\frac{\partial}{\partial\mu^2} + \underbrace{2\mu^2\,\frac{\partial\lambda}{\partial\mu^2}\Big|_{\lambda_0,m_0}}_{\sigma}\frac{\partial}{\partial\lambda} - \frac{n}{2}\underbrace{2\mu^2\,\frac{\partial}{\partial\mu^2}\log Z_2\Big|_{\lambda_0,m_0}}_{2\tau}\right\}\Gamma^{(n)} = 0 \tag{24.5}$$

$\sigma$ and $\tau$ are finite when the regulator goes to zero and thus $\lambda_0$ and $m_0 \to \infty$. Comparing eq. (24.5) to the CS equation (24.1) with eq. (24.4) gives $\beta = \sigma + \ldots$ and $\gamma = \tau + \ldots$ up to terms proportional to $m^2$, which are suppressed by $m^2/p^2$ in $\Gamma^{(n)}$-solutions.

Usually, $\beta$ and $\gamma$ are calculated as follows (this is general for quantum field theories):

- calculate the renormalization pieces (in $\Phi^4$-theory, these are the 2- and 4-point functions) to some order in perturbation theory
- write down the renormalization group equations for these 1PI amplitudes and obtain a set of equations for $\beta$ and $\gamma$

For $\Phi^4$-theory, one obtains to second order in $\lambda$:

$$\gamma = \frac{\lambda^2}{8\pi^2} + \mathcal{O}(\lambda^3); \qquad \beta = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)$$

(**Exercise**: check this.)

## 24.3 Solutions to the RG equations

The RG equation

$$\left[2\mu^2\frac{\partial}{\partial\mu^2} + \beta(\lambda)\frac{\partial}{\partial\lambda} - n\gamma(\lambda)\right]\Gamma^{(n)}_{\text{as}}(p_1, \ldots, p_n) = 0 \tag{24.6}$$

has a solution

$$\Gamma^{(n)}_{\text{as}}(p_1, \ldots, p_n) = s^{(4-n)/2} f\left(\frac{p_i p_j}{s}, \log\frac{\mu^2}{s}, \lambda\right) \tag{24.7}$$

which can be found by dimensional analysis. Here, $s$ has taken the place of $p^2$ in section 24.1, and we have neglected $m$, since we assume $p^2 \gg m^2$. So,

$$\boxed{\left[\frac{\partial}{\partial(\log\sqrt{\mu^2/s})} + \beta(\lambda)\frac{\partial}{\partial\lambda}\right]\Gamma^{(n)} = n\gamma(\lambda)\Gamma^{(n)}} \tag{24.8}$$

### Bacteria analogue

A nice analogue due to Coleman interprets these equations as describing the development of a bacteria population in a moving fluid, whose growth depends on its exposure to infalling light. If one makes the following identifications:

$$\begin{aligned}
\log\sqrt{\frac{\mu^2}{s}} &\sim t \quad \text{(time)}\\
\lambda &\sim x \quad \text{(position)}\\
\beta(\lambda) &\sim v(x) \quad \text{(fluid velocity)}\\
n\gamma(\lambda) &\sim L(x) \quad \text{(space dependent illumination)}\\
\Gamma^{(n)} &\sim \rho \quad \text{(bacteria density, moving with fluid)}
\end{aligned}$$

eqs. (24.6) become

$$\underbrace{\frac{\partial\rho(x,t)}{\partial t} + v(x)\frac{\partial\rho(x,t)}{\partial x}}_{\frac{\mathrm{d}\rho}{\mathrm{d}t}\text{in comoving system}} = L(x)\rho(x,t) \tag{24.9}$$

Let us follow a small element, being at $x$ at time $t$, backwards in time: we define its position at time $t = 0$ as $\bar{x}(x,t)$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(x,t) = -v(\bar{x}) \quad \text{with} \quad \bar{x}(x,0) = x$$

Using this relation, the development of the local density $\rho(x,t)$ can be rewritten as follows:

$$\rho(x,t) = f_{\text{initial}}(\bar{x}(x,t)) \exp\left\{\int_0^t dt' L(\bar{x}(x,t'))\right\} = \exp\left\{\int_{\bar{x}(x,t)}^x dx' \frac{L(x')}{v(x')}\right\}$$

where $f_{\text{initial}}(\bar{x}(x,t))$ represents the density in the comoving volume element at $t = 0$.

Translating back to our quantum field theoretical interpretation of the equations, we get

$$\lambda'(\lambda,0) = \lambda \quad \text{and} \boxed{\frac{\mathrm{d}\lambda'(\lambda,t)}{\mathrm{d}t} = -\beta(\lambda')} \tag{24.10}$$

Our graph expression becomes

$$\Gamma^{(n)}_{\text{as}} = s^{(4-n)/2} f^{(n)}\left(\frac{p_i p_j}{s}, 0, \lambda'\left(\lambda, \frac{1}{2}\log\frac{\mu^2}{s}\right)\right) \exp\left\{n\int_0^{\frac{1}{2}\log\frac{\mu^2}{s}} dt\gamma(\lambda'(\lambda,t))\right\}$$

Changing variables from $\lambda, \mu^2$ to $\lambda', \mu^2 = s$, and setting $\beta = 0$ in eq. (24.10), we have $\lambda' = \lambda$ =const and

$$\Gamma^{(n)}_{\text{as}} = s^{(4-n)/2}\left(\frac{s}{\mu^2}\right)^{-n\gamma(\lambda)/2} f^{(n)}\left(\frac{p_i p_j}{s}, 0, \lambda\right)$$

Of course, with our present perturbative methods, we can only obtain $\beta(\lambda)$ in the perturbative regime, where $\lambda$ is small. However, the renormalization group equation does not contain a direct relation to perturbation theory, so we could simply try to apply it more generally, in spite of its perturbative derivation. We can identify two regions with regard to its general behavior (see fig. 24.1):

(i) $0 \le \lambda \le \lambda_1$: here, $\beta(\lambda) > 0$. $\lambda'$ increases monotonically with $\bar{t} = -t$, $t = \frac{1}{2}\log\frac{\mu^2}{s}$, but cannot go beyond $\lambda_1$, since there, $\beta(\lambda_1) = 0$. Thus,

$$\lim_{\bar{t}\to\infty(s\to\infty)} \lambda'(\lambda,\bar{t}) = \lambda_1$$

$\lambda_1$ is an *ultraviolet fixed point.*

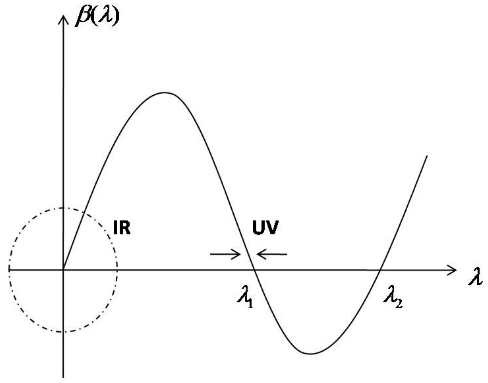



Figure 24.1: Application of the RG equation outside the perturbative regime

(ii) $\lambda_1 \leq \lambda \leq \lambda_2$: here, $\beta(\lambda) < 0$. $\lambda'$ decreases monotonically with $\bar{t} = -t$ to the UV fixed point $\lambda_1$.

For $\bar{t} = -t \to -\infty$, i.e. $s \to 0$, in case (i), $\lambda'$ goes to zero, and in case (ii) to $\lambda_2$: these are *infrared* fixed points. This is only strictly true for *massless theories*. In QCD, $\beta$ is negative for small $g^2$, and there is an ultraviolet fixed point at $g^2 = 0$ (asymptotic freedom).

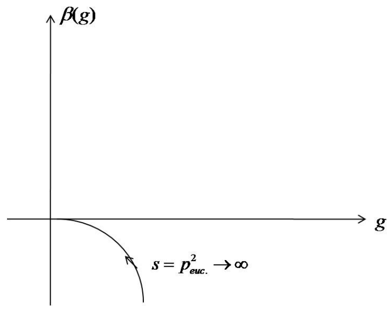



Figure 24.2: QCD: asymptotic freedom

## 24.4 Wilson renormalization group

The Wilson renormalization group is a huge subject, and can only be touched upon here, but is indispensable for a modern understanding of QFT.

So far, we have concentrated on renormalizalble local quantum field theories where one can hide the divergencies at small distances, or large momenta, in the redefinition of a finite number of couplings, and afterwards take the cut-off (regulator) to infinity. However, we do not know the ultimate theory at small distances or high energies, and thus local Lagrangians are only a model construction (with the possible exception of pure Yang-Mills theory, where asymptotic freedom and dimensional transmutation together make the high-energy limit tractable). Now, arriving from statistical mechanics at QFT, it is clear that one has to start with an effective description, i.e. Lagrangian, at some cut-off scale $\Lambda$, and that one should never integrate over the unphysical domain of momenta exceeding $\Lambda$. This, of course, presents dangers: one might lose the symmetries encoded in local QFT, and can only *hope* that they will be redetected in the IR (where, in the case of solid-state physics, one does not see the lattice any longer).

In the Wilson approach to renormalization, one restricts the partition function to momenta below a cut-off $\Lambda$. Performing a unitary transformation which rewrites the path integral in terms of the momentum modes, we get

$$Z^\Lambda = \int_{|k|\leq\Lambda} \mathcal{D}\Phi(k) \exp\left\{ - \int dx \mathcal{L}^\Lambda \right\}$$

where $\mathcal{L}^\Lambda$ is defined by

$$\tilde{\mathcal{L}}_0^\Lambda = \frac{1}{2} \underbrace{\tilde{\Phi}^*(k)}_{\tilde{\Phi}(-k)} k^2 \tilde{\Phi}(k)$$

and $k^2 \leq \Lambda^2$ in the Euclidean form. Now, we perform the path integration in steps:

- we divide $\Phi$ into $\Phi = \Phi_< + \Phi_>$, where $\Phi_<$ contains the Fourier components $|k| \leq b\Lambda$, for $b < 1$, and $\Phi_>$ contains the Fourier components $b\Lambda < |k| \leq \Lambda$
- we integrate only $\Phi_>$ in the path integral (after decomposing also the propagators into $P_>$ and $P_<$, etc.):

$$\int_{b\Lambda<|k|\leq\Lambda} \mathcal{D}\Phi_> \exp\left\{ - \int cd\mathcal{L}^\Lambda \right\} = \exp\left\{ - \int dx \mathcal{L}_{\text{eff}}^{b\Lambda} \right\}$$

- we rename $\Phi_< \to \Phi$, so

$$Z^\Lambda = \int_{|k|\leq b\Lambda} \mathcal{D}\Phi \exp\left\{ - \int dx \mathcal{L}_{\text{eff}}^{b\Lambda} \right\}$$

- we *rescale* $k \to k' = k/b$ and $x \to x' = xb$, so $|k'| < \Lambda$, and then we rescale the field $\Phi$, yielding the original path integral, but with a transformed Lagrangian

For example, in $\Phi^4$-theory, we would have:

$$\mathcal{L}^{\Phi^4} = \frac{1}{2}(\partial_\mu \Phi_< + \partial_\mu \Phi_>)^2 + \frac{1}{2}m^2(\Phi_< + \Phi_>)^2 + \frac{\lambda}{4!}(\Phi_< + \Phi_>)^4 =$$
$$\mathcal{L}_<^{\Phi^4} + \mathcal{L}_>^{\Phi^4} + \lambda\left(\frac{1}{3!}\Phi_<^3\Phi_> + \frac{1}{4}\Phi_<^2\Phi_>^2 + \frac{1}{3!}\Phi_<\Phi_>^3\right)$$

(The $\Phi_<\Phi_>$-terms drop out after integrating $\Phi_>$.) Now we do the $\Phi_>$-integration considering the $\lambda$-terms as a perturbation, which allows us to use the Feynman diagram method. Then, $\int dx \mathcal{L}_{\text{eff}}^{b\Lambda}$ is the sum of all connected graphs, like those in fig. 24.3.

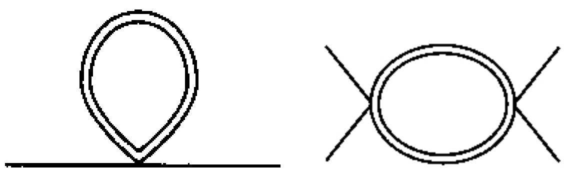


Figure 24.3: Connected graphs in $\int dx \mathcal{L}_{\text{eff}}^{b\Lambda}$

All kinds of new terms appear in $\mathcal{L}_{\text{eff}}^{b\Lambda}$:

$$\int d^D x \mathcal{L}_{\text{eff}} = \int d^D x \left[\frac{1}{2}(1+\Delta Z)(\partial_\mu \Phi_<)^2 + \frac{1}{2}(m^2 + \Delta m^2)\Phi_<^2 + \right.$$
$$\left. \frac{1}{4}(\lambda + \Delta\lambda)\Phi_<^4 + \Delta c(\partial_\mu \Phi_<)^4 + \Delta d \Phi_<^6 + \ldots\right] =$$
$$\int d^D x' b^{-D}\left[\frac{1}{2}(1+\Delta Z)b^2(\partial'_\mu \Phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\Phi^2 + \right.$$
$$\left. \frac{1}{4}(\lambda + \Delta\lambda)\Phi^4 + \Delta c b^4(\partial'_\mu \Phi)^4 + \Delta d \Phi^6 + \ldots\right]$$

Rescaling $\Phi \to \Phi' = \left[b^{2-D}(1+\Delta Z)\right]^{1/2}\Phi$,

$$\int d^D x \mathcal{L}_{\text{eff}} = \int d^D x' \left[\frac{1}{2}(\partial'_\mu \Phi')^2 + \frac{1}{2}m'^2\Phi'^2 + \frac{1}{4}\lambda'\Phi'^4 + c'(\partial'_\mu \Phi')^4 + d'\Phi'^6 + \ldots\right]$$

with

$$m'^2 = (m^2 + \Delta m^2)(1+\Delta Z)^{-1}b^{-2}; \quad \lambda' = (\lambda + \Delta\lambda)(1+\Delta Z)^{-2}b^{D-4}$$
$$c' = (c + \Delta c)(1+\Delta Z)^{-2}b^D; \quad d' = (d + \Delta D)(1+\Delta Z)^{-3}b^{2D-6}$$

The above procedure can be repeated. For $b$ close to unity, we obtain a "continuous" flow in $\mathcal{L}_{\text{eff}}^{b^n\Lambda}$.

**Example**

The above procedure yields for the second graph in fig. 24.3 the expression

$$
\begin{aligned}
&-\frac{1}{4!}\int d^D x \rho \Phi^4 \qquad \text{where} \\
\rho = &-4!\frac{2}{2!}\left(\frac{\lambda}{4}\right)^2 \int_{b\Lambda \le |k| \le \Lambda} \frac{d^D k}{(2\pi)^D}\left(\frac{1}{k^2}\right)^2 = \\
&\frac{-3\lambda^2}{(4\pi)^{D/2}\Gamma(D/2)} \frac{1-b^{D-4}}{D-4}\Lambda^{D-4} \\
&\stackrel{D\to 4}{\to} -\frac{3\lambda^2}{16\pi^2}\log\frac{1}{b}
\end{aligned}
$$

which is a finite correction

**Note**

If we now calculate an amplitude with outer momenta $k_{\text{out}}$, we can calculate with the original $Z^\Lambda$, which contains $\mathcal{L}^\Lambda$ and a source term $\Phi J$, or with the new form containing $\mathcal{L}^{b^n\Lambda}$. The latter is more practical, since all the higher momenta $k > k_{\text{out}}$ have already been integrated out ($b^n\Lambda \approx k_{\text{out}}$).

Staying in $\Phi^4$-theory, it is particularly simple to start in the vincinity of $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \Phi)^2$. Then,

$$
m'^2 = m^2 b^{-2}; \quad \lambda' = \lambda b^{D-4}; \quad c' = cb^D; \quad d' = db^{2D-6}
$$

Here, 'in the vincinity of $\mathcal{L}_0$' means that we allow all interactions with coefficients small enough to allow us to perform the integrations $b\Lambda \le k \le \Lambda$ perturbatively.

As we iterate the procedure described above, negative powers of $b < 1$ grow, positive powers fall off, and zero powers naturally stay constant. This has consequences for the role of the operators multiplied by these powers; they are classified as follows:

| | |
|---|---|
| negative powers of $b$ (growing) | *relevant* operators |
| positive powers of $b$ (falling off) | *irrelevant* operators |
| zero powers of $b$ (constant) | *marginal* operators |

All irrelevant operators die out as a consequence of the *renormalization group flux.* (As an aside, note that strictly speaking, the renormalization group is not a group, since the inverse of integrating out does not exist.)

In our example close to $\mathcal{L}_0$, we are left with a renormalizable Lagrangian for large $n$ (step number). Be careful, however: for $m'^2 = m^2 b^{-2n} \approx \Lambda^2$, everything is integrated out, and the procedure *stops.* For $m^2 \ll \Lambda^2$, one has many steps.

**Example**

Peskin & Schröder give a nice example: they consider $\Phi^4$-theory in $D > 4$, $D = 4$ and $D < 4$ dimensions, near $\lambda = 0, m = 0$.

If $D > 4$, the $\Phi^4$-term is irrelevant, and the only relevant term is the one proportional to $m^2$; see fig. 24.4. Figures 24.5 and 24.6 show the cases where $D = 4$ (the $\Phi^4$-term is marginal) and $D < 4$ (the $\Phi^4$-term is relevant).

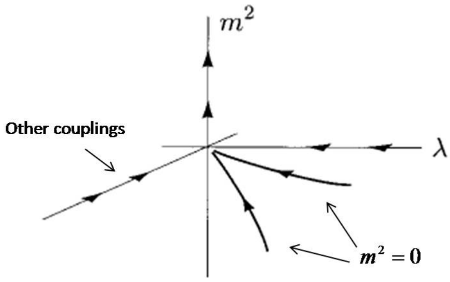



Figure 24.4: $D > 4$: only $m^2$ is relevant

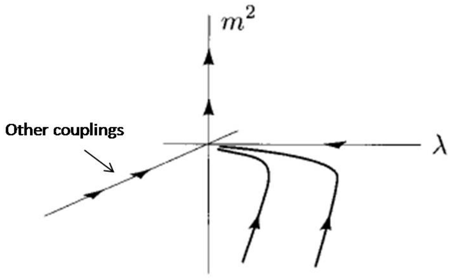



Figure 24.5: $D > 4$: $\Phi^4$ is marginal

The renormalizable piece of the Lagrangian dominates in the IR, and if the couplings are small, it can be treated perturbatively with the Callan-Szymanzik equation (or renormalization group equation). If the couplings are big, it can be treated with the Wilson procedure, truncating the irrelevant part. The latter is hard to judge, however, if one is far away from perturbation theory.

**Note**

Wilson renormalization and the renormalization group are very important in the theory of critical phenomena, second order phase transitions in statistical

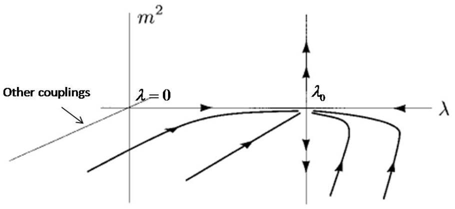



Figure 24.6: $D > 4$: $\Phi^4$ is relevant

mechanics (see e.g. Wegner's work). There, large correlations are important, and one has predictions for the IR.

In elementary particle physics, the aspect that in the IR we have renormalizable effective theories is very important as well, but there also is the urgent questions how to write down QFT starting from a *classical theory* in the *continuum*. This correspondence only makes sense if we have a weakly interacting theory at small distances, i.e. in the UV; in other words, we need asymptotic freedom, as in QCD. In its usual formulation from Maxwell theory, QED has an IR fixed point. It were preferable to have it embedded in a non-abelian gauge theory, a so-called *Grand Unified Theory*, which in turn might be an effective IR theory itself, emanating from some more fundamental theory.

## 24.5 Renormalization group flow for the effective action

This topic is also known as the *exact renormalization group equations*, and has been treated by C. Wetterich and U. Ellwanger, amongst others. C. Bagnuls and C. Bervillier have published a review article, **hep-th/0002034** (their notation differs from the one used here by a minus sign: $W \to -W$, $J \to -J$, etc.). Also see C. Wetterich's lectures.

Recall the effective action, in Euclidean form:

$$Z(J) = \exp(-W(J)) = \int \mathcal{D}\Phi \exp\left\{-\left(S(\Phi) + \int dx J(x)\Phi(x)\right)\right\}$$

(The notation $J * \Phi$ is also used for $\int dx J(x)\Phi(x)$.) Performing a Legendre transform,

$$\begin{aligned}\frac{\delta W}{\delta J(x)} &= \varphi(x) \\ \Gamma(\varphi) &= W(J(\varphi)) - J * \varphi\end{aligned}$$

where $\Gamma(\varphi)$ is the *effective action*, which in perturbation theory corresponds to the 1PI diagrams. Staying with $W(J)$, the free energy, one can calculate $\langle\Phi(x)\rangle$ as well as a *mean field* $\Phi_k(x)$ given by

$$\Phi_k(x) = \frac{1}{V_k}\int_{V_k} d^D x \langle\Phi(x)\rangle$$

over some volume $V_k \propto k^{-D}$, introducing a cut-off $k$ in momentum space. This new field $\Phi_k(x)$ has a new action with $\mathcal{L}_{\text{eff}}^{(k)}$ with scale $k$. This scale can be changed by integrating over larger and larger volumina - this is the Wilson / Kadanoff approach we have discussed before. Already in this approach, the choice of the cut-off function is important, if one wants to perform (part of) the procedure analytically. For example, one could choose a "filter"

$$R_k(q^2) = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$

If $q^2 \ll k^2$, this functions like a mass term $Z_k k^2$; if $q^2 \gg k^2$, it goes like $e^{-q^2/k^2}$. The cut-off function is introduced into the action via an additional term

$$\Delta_k S = \frac{1}{2}\int \frac{d^D q}{(2\pi)^D}\Phi^*(q) R_k(q^2)\Phi(q) \quad \text{with real } \Phi$$

Here, $k$ is an IR-cut-off; one can also introduce a UV-cut-off $\Lambda$.

All of this becomes much more transparent if we consider the effective mean field action $\Gamma_k(\varphi)$:

$$\boxed{\exp(-W_k(J)) = \int \mathcal{D}\Phi \exp\left\{-(S(\Phi) + J * \Phi + \Delta_k S(\Phi))\right\}} \tag{24.11}$$

and

$$\frac{\delta W_k}{\delta J} = \varphi$$
$$\tilde{\Gamma}_k(\varphi) = W_k(J(\varphi)) - J * \varphi$$

We have to correct for the added term $\Delta_k S$:

$$\Gamma_k(\varphi) = \tilde{\Gamma}_k(\varphi) - \Delta_k S(\varphi)$$

Then, we have $\Gamma_k \to S_\Lambda$ for $k \to \Lambda$ (remember, $k \leq \Lambda$), and $\Gamma_k \to \Gamma$ for $k \to 0$. Now, since $k$ is a continuous parameter, it is very natural to write down a differential relation, i.e. to calculate

$$k\frac{\partial}{\partial k}\Gamma_k = \partial_t \Gamma_k = \partial_t \tilde{\Gamma}_k - \partial_t(\Delta_k S)$$

Then,

$$\text{“d}_t\text{”}\tilde{\Gamma}_k = \partial_t W_k = \langle \partial_t \Delta_k S \rangle = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \partial_t R_k(q^2) \langle \Phi^*(q)\Phi(q) \rangle =$$
$$\frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \partial_t R_k(q^2) \left\{ G(q) + \varphi^*(q)\varphi(q) \right\}$$

The last term is subtracted to obtain $\partial_t \Gamma_k$ (here in operator formulation):

$$\boxed{\partial_t \Gamma_k(\varphi) = \frac{1}{2} \mathrm{tr} \partial_t \mathcal{R} G} \tag{24.12}$$

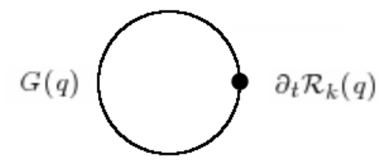
with

$$\mathcal{R}(q, q') = R_k(q)(2\pi)^D \delta^D(q - q') \quad \text{and}$$
$$G(q, q') = \langle \Phi^*(q)\Phi(q') \rangle_c = \frac{\delta^2 W}{\delta J(q) \delta J^*(q')} =$$
$$(\tilde{\Gamma}_k^{(2)})^{-1}(q, q') = (\Gamma_k^{(2)} + \mathcal{R})^{-1}$$

with $\Gamma_k^{(2)}$ the inverse propagator in the background $\varphi$. Thus, we obtain for $\partial_t \Gamma_k(\varphi)$:

$$\boxed{\partial_t \Gamma_k(\varphi) = \frac{1}{2} \mathrm{tr}(\partial_t \mathcal{R}(\Gamma_k^{(2)}(\varphi) + \mathcal{R})^{-1})} \tag{24.13}$$

This is known as the *exact (nonperturbative) renormalization flux equation.*




$\Gamma_k$ now is the most general effective action, fixed at some $k = \Lambda$ and then developing to $k = 0$. The RG flow equation relates the various 1PI $n$-point functions. For example,

$$\partial_t \Gamma_k^{(2)}{}_{(q)} \Big|_{\varphi=0} = \partial_t \left. \frac{\partial^2 \Gamma_k(\varphi)}{\partial \varphi^*(q) \partial \varphi(q)} \right|_{\varphi=0} =$$
$$\frac{\partial}{\partial \varphi^*(q)} \frac{1}{2} \mathrm{tr} \left[ \partial_t \mathcal{R} (-1) (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \frac{\partial \Gamma_k^{(2)}}{\partial \varphi(q)} (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \right]_{\varphi=0} =$$
$$\mathrm{tr} \left[ \partial_t \mathcal{R} \frac{2(-1)^2}{2} (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \frac{\partial \Gamma_k^{(2)}}{\partial \varphi^*(q)} (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \frac{\partial \Gamma_k^{(2)}}{\partial \varphi(q)} (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \right] -$$
$$\frac{1}{2} \mathrm{tr} \left[ \partial_t \mathcal{R} (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \frac{\partial^2 \Gamma_k^{(2)}}{\partial \varphi^*(q) \partial \varphi(q)} (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \right]_{\varphi=0}$$

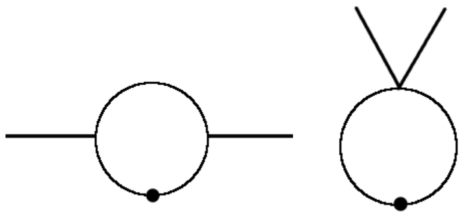


Figure 24.7: 3- and 4-point functions

This corresponds to the following diagrams:

Similarly, one can derive equations for $\partial_t \Gamma_k^{(3)}$, $\partial_t \Gamma_k^{(4)}$, etcetera, yielding an infinite set of equations. To avoid this, we need to truncate the series at some point. In the perturbative regime, one can easily identify which pieces are relevant and marginal, and which ones can be neglected. In the case of large couplings, i.e. the non-perturbative regime, the right choice of the relevant pieces is difficult, and has to be found by trial and error, educated guessing, or prejudices. The gauge covariant formulation of the program is quite demanding.

## 24.6 Lattice gauge theory

For literature on lattice gauge theory, see I. Montvay & G. Münster (CUP), H. Rothe, and J. Smit. The strong coupling regime of QCD requires appropriate approximation methods. Discretization of the degrees of freedom on a lattice should not destroy *gauge invariance* for physical results, like hadron masses and cross sections. Our original motivation for gauge fixing came from our wish for representations on the gauge orbits, in order to avoid zero modes in the path integration, which would lead to infinities. On a finite lattice, however, everything is finite, and thus a completely gauge invariant formalism is desirable - this is the Wegner-Wilson approach to lattice gauge theory. It works as follows:

(i) An object like

$$\bar{\Psi}^B(x') \left( \mathcal{P} \exp \left\{ i \int_x^{x'} \mathbb{A}_\mu dx^\mu \right\} \right)_B^{\;A} \Psi_A(x)$$

is *gauge invariant*: $\Psi$ and $\bar{\Psi}$ transform in the fundamental and anti-fundamental representations of $SU(3)$, respectively:

$$\begin{aligned} \Psi'_A(x) &= \Lambda_A^B(x)\Psi_B(x) \\ \bar{\Psi}'^B(x') &= \bar{\Psi}^A(x')(\Lambda^\dagger(x'))_A^{\cdot B} \end{aligned}$$

with a unitary $\Lambda$ ($\Lambda\Lambda^\dagger = 1$) gauge transformation. The connection (with "path ordering" $\mathcal{P}$), also known as *Schwinger string*, in the adjoint representation:

$$U_A^B(x,x') = \left(\mathcal{P}\exp\left\{i\int_x^{x'} d\bar{x}^\mu \mathbb{A}_\mu(\bar{x})\right\}\right)_A^{\cdot B}$$

transforms as

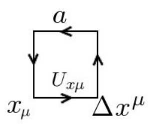
$$\boxed{U'(x',x) = \Lambda(x')U(x',x)\Lambda^\dagger(x)} \tag{24.14}$$

The proof (albeit somewhat sketchy), goes as follows: the path ordering ($\mathcal{P}$) essentially means breaking down the path into infinitesimal pieces (compare $e^x - \lim_{n\to\infty}(1+x/n)^n$) in a path ordered manner. Then,

$$\bar{\Psi}(x')\left(\exp\left\{i\int_x^{x'} dx^\mu \mathbb{A}_\mu\right\}\right)\Psi(x) \approx \bar{\Psi}(x')(\mathbb{1} + \underbrace{i\mathbb{A}_\mu(x')\Delta x^\mu - \Delta x^\mu\partial_\mu}_{-\Delta x^\mu \mathcal{D}_\mu})\Psi(x')$$

with $\mathcal{D}'_\mu = \Lambda\mathcal{D}_\mu\Lambda^\dagger$, which is a tensor transformation in $n$-dimensional space. Hence, $\exp\left\{i\int_x^{x'} dx^\mu \mathbb{A}_\mu\right\}$ is a gauge covariant expression.

(ii) Now introduce such $U$'s on a *square lattice*: the $U$'s will be on the links, the edges between lattice points, and the $\Psi$'s will be on lattice points ("sides").

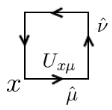



$|\Delta x^\mu| = a$

(iii) Finally, consider a "plaquette" in pure Yang-Mills theory:

$$U_p = U(x, x+a\hat{\nu})U(x+a\hat{\nu}, x+a\hat{\mu}+a\hat{\nu})U(x+a\hat{\mu}+a\hat{\nu}, x+a\hat{\nu})U(x+a\hat{\mu}, x) = U_{x;\mu,\nu}$$

with $x$ on the lattice:




The Wilson lattice action is written in terms of these gauge invariant plaquettes:

$$S = \beta\sum_p\left(1 - \frac{1}{3}\Re\,\mathrm{tr}\,U_p\right) = -\beta\sum_p\left(\frac{1}{2\,\mathrm{tr}\,\mathbb{1}}(\mathrm{tr}\,U_p + \mathrm{tr}\,U_p^{-1}) - 1\right)$$

where the factor of 1/3 in the first line is actually $1/(\mathrm{tr}\mathbb{1})$, which is 1/3 in the case of $SU(3)$. Also note that $S = 0$ for $U = \mathbb{1}$.

Using the Campbell-Baker-Hausdorff formula $e^x e^y = e^{x+y} e^{\frac{1}{2}[x,y} \cdots$, one obtains

$$
\begin{aligned}
U_{x,\mu\nu} &= \exp(-a^2 \mathbb{G}_{\mu\nu}) \\
\mathbb{G}_{\mu\nu} &= \mathbb{F}_{\mu\nu} + \mathcal{O}(a) \\
\mathbb{F}_{\mu\nu} &= \Delta_\mu \mathbb{A}_\nu - \Delta_\nu \mathbb{A}_\mu + [\mathbb{A}_\mu(x), \mathbb{A}_\nu(x)] \\
\mathrm{tr}(U_p + U_p^{-1}) &= 2\,\mathrm{tr}\,\mathbb{1} + a^4 \mathrm{tr}\,\mathbb{F}_{\mu\nu}^2 + \text{higher orders in } a
\end{aligned}
$$

(with $\Delta_\mu \mathbb{A}_\nu = \mathbb{A}_\nu(x + a\hat{\mu}) - \mathbb{A}_\nu(x)$) from which we conclude

$$\beta/2\mathrm{tr}\mathbb{1} = g^{-2}$$

if we want to obtain the continuum action of QCD for $a \to 0$. The path integral is over all $U$ on all links: these are group integrations with the (invariant) Haar measure

$$\int \mathcal{D}U \exp(-S(U))$$

This lattice action can be used to calculate corelation functions (propagators) of operators composed of $U$'s, which should perferably be gauge invariant objects, like plaquette traces. For small $\beta$ and large $g^2$ (the so-called "high temperature expansion"), one can do perturbative calculations and read off masses. Starting with some lattice with spacing $a$ with $N^4$ lattice points, one can refine the lattice to one with $(2N)^4$ points, with spacing $a/2$; thus, the lattice covers the same physical volume, and one should obtain the "same" results (e.g. hadronization). Of course, this also holds if one repeats this step $n$ times, and scales to a lattice spacing of $a/2^n$. For this to work, the gauge coupling should "run", i.e. depend on the lattice spacing: $g = g(a/2^n)$. For very fine lattices, we expect $g(\mu = 2^n/a)$ to become small, and we should recover the results of perturbative QCD. Of course, one can do the usual perturbation theory with a lattice regulator, and compare with such calculations (breaking Lorentz invariance because of the lattice).

The form of the QCD Lagrangian is taken close to the UV fixed point $g \to 0$, and indeed, one should run the renormalization group á la Wilson to *integrate out* lattice points going to the IR. There, we do *not* expect the lattice action we started from - in the strong coupling limit, there are indeed further terms in the action.

Adding fermions (quarks) to the theory allows for string-like hadronic states: fig. 24.8 shows a baryon on the left, and a (fluctuating) meson to the right. In the strong coupling limit, the "string" becomes more and more rigid, whereas it is fluctuating strongly at small distances (i.e. small couplings); this yields a Coulomb-like behaviour. The treatment of fermions on the lattice requires particular effort, because chiral invariance is easily violated (see literature).

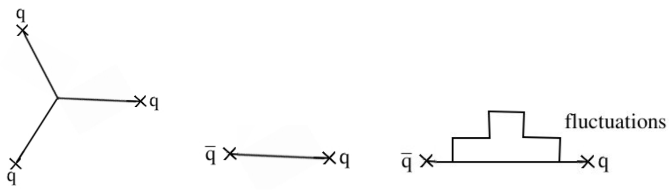



Figure 24.8: Hadronic states

## 24.7 Other topics

Other important aspects of quantum field theory, that are beyond the scope of these lectures, are (amongst others):

- Quantum anomalies: the regulator needed for quantization violates a symmetry of the classical Lagrangian
- Non-perturbative physics: instantons, solitons
- Supersymmetry: we have already seen the BRST symmetry; general supersymmetry, which might be realized in nature, relates fermions and bosons by a Grassmann-type symmetry, such that fermionic and bosonic determinants cancel - this makes the theory more classical
- Resummation of IR logarithms; thermal QFT
- Spontaneous symmetry breaking: there is more to it than we have seen here
- QFT in curved spaces