Finite temperature QCD in equilibrium: lattice Monte Carlo, perturbation theory and AdS/QCD.

K. Kajantie
Helsinki Institute of Physics

Heidelberg, 15-19 March 2010

We want to **solve** QCD

$$\mathcal{L}_{\text{QCD}} = \frac{1}{4g^2} \sum_{a=1}^{N_c^2-1} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{i=1}^{N_f} \bar{\psi}_i [\gamma_\mu D_\mu + m_i] \psi_i$$

and related theories

Contents

1. Lattice and finite T phenomena:

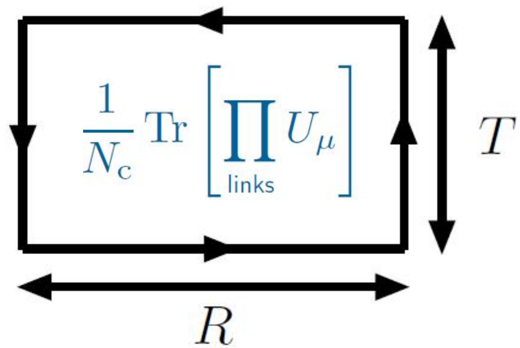
1. For the equation of state, evaluate the integral

You always have the confining magnetic sector!

$$Z(T,V) = e^{p(T)\frac{V}{T}} = \int \mathcal{D}[A\bar{\psi}\psi]e^{-\int_0^{1/T} d\tau d^3x\, \mathcal{L}_{\text{QCD}}}$$

2. Spatial string tension(T)

Expectation value of



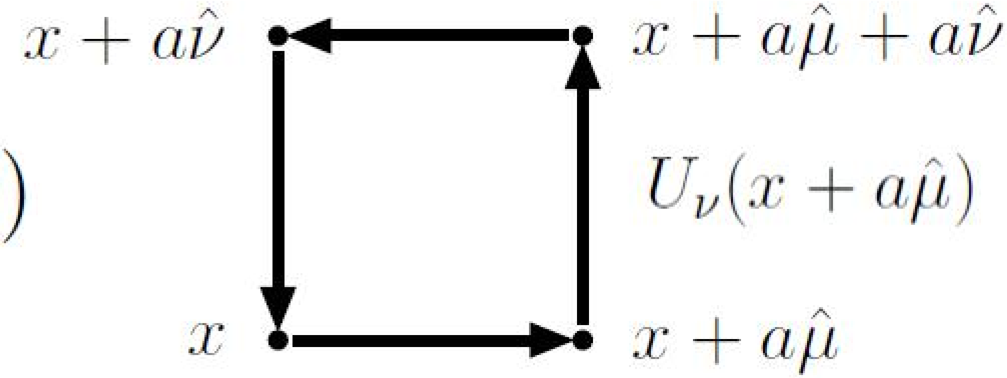
with path in spatial directions

3. Polyakov line $\frac{1}{N_c}\text{Tr}\left[\prod_{\text{links}} U_0\right]$

with path in τ direction

Lattice: $N_t \cdot N_s^3$ $\quad \frac{1}{T} = N_t a \ll N_s a = V^{1/3}$

$P_{\mu\nu}(x)$



$$U_\mu(x) = e^{igaA_\mu(x)}$$

G(x), ψ(x) on sites, gauge fields on links!

Gauge transfo $\quad U'_\mu(x) = G(x)U_\mu(x)G^{-1}(x + a\hat{\mu})$

Tr[loop] = gauge invariant

$$S_E \equiv \frac{1}{g_0^2} \sum_x \sum_{\mu,\nu=0}^{3} \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}(x)\right] \approx a^4 \sum_x \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

On the lattice one Monte Carloes expectation values = derivatives of logZ

$$\langle I \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \, I[U] e^{-S_E[U]}$$

$4 N_t N_s^3 (N_c^2 - 1) \sim 10^7$ dim integral

Normalisation cancels!

$$\langle \text{gauge noninvariant} \rangle = 0$$

Fermions

$$a^4 \sum_{x,y} \bar{\psi}(x) \left[D(x,y) + M\delta_{x,y}\right] \psi(y) \;\; -\frac{r}{2} \sum_x a^5 \bar{\psi}(x) \Delta_\mu \Delta_\mu^* \psi(x)$$

Wilson term

Grassman variables integrated over:

$$\mathcal{Z} = \int \mathcal{D}U_\mu \; \mathsf{Det}[D+M] \exp\left\{-S_E^{(\text{gluons})}\right\}$$

$10^7 \cdot 10^7$ sparse matrix

$$\langle \psi(x)\bar{\psi}(y) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U_\mu \; \mathsf{Det}[D+M][D+M]^{-1}(x,y) e^{-S_E^{(\text{gluons})}}$$

Long non-ending story, chiral symmetry, overlap fermions, domain wall fermions, connection to analytic formulas of chiral perturbation theory

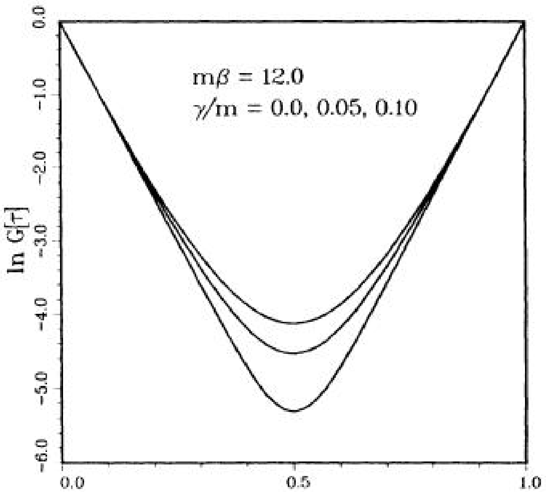
Limitation: lattice OK for Euclidian, static phenomena!
Not even stationary, like transport coefficients

$$G_\beta(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \rho(\omega) \frac{\exp(-\omega\tau)}{1-\exp(-\beta\omega)}$$

$$\int d^3x \, \langle \pi_{kl}(\tau, \mathbf{x}) \pi_{kl}(0, \mathbf{0}) \rangle \qquad \rho(\omega) \sim \eta\omega + ..$$

?

Ill-posed problem!!



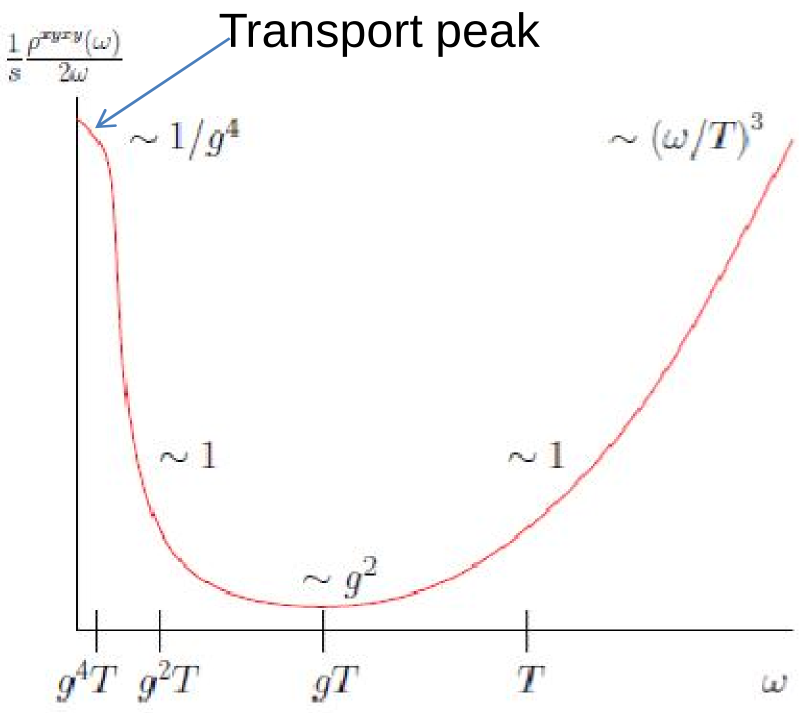
Correlator determined in a discrete set of ~ Nτ points

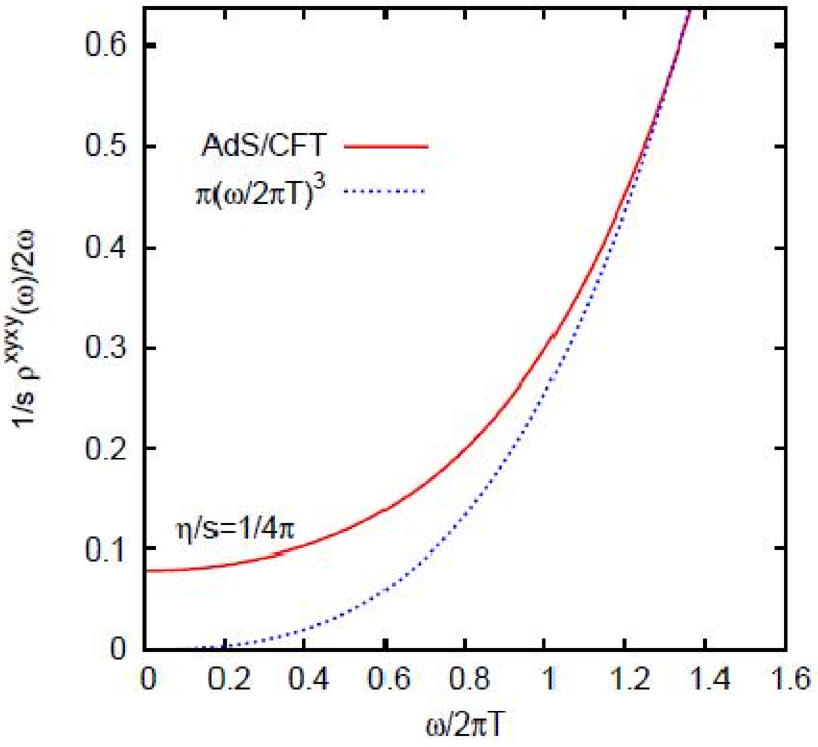
Parametrize ρ(ω) by physics and fit parameters

Karsch-Wyld, PRD 1987

Small-ω structure of ρ(ω) is complicated in weak coupling (many different scales), simpler in strong coupling (T is the only scale) Aarts-Resco hep-lat/0110145

Meyer 0907.4095





Schafer-Teaney 0904.3107

Lattice and p(T)

What expectation value gives the EoS? Since

$$\log Z = \frac{p(T)}{T} V = \log \int \mathcal{D}U e^{-\beta(a) S_\Box(U)} \quad \frac{1}{T} \sim a \quad V \sim a^3$$

$$\frac{-1}{VT^3}\, a \frac{d \log Z}{da} = \frac{\epsilon - 3p}{T^4} = T \frac{\partial}{\partial T} \frac{p(T)}{T^4}$$

$$= \frac{N_t^3}{N_s^3}\, a \beta'(a) \langle S_\Box \rangle_{(T)-(T=0)} \qquad a \frac{d(ma)}{da} \sum_x \langle \bar{\psi} \psi \rangle$$

So ”just” determine the expectation value of the plaquette action times lattice beta function!!

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{s}{T s'(T)} \qquad T \frac{\partial}{\partial T} \frac{s}{T^3} = \frac{s}{T^3} \left(\frac{1}{c_s^2} - 3 \right)$$

Cherman-Cohen-Nellore 0905.0903

Physics is in decimals:

$$\frac{\epsilon - 3p}{T^4} = N_t^4 \, a\frac{d}{da}\frac{2N_c}{g^2(a)}\left[\langle\frac{S_\Box}{N_t N_s^3}\rangle_{N_t N_s^3} - \langle\frac{S_\Box}{N_s^4}\rangle_{N_s^4}\right]$$

$\mathcal{O}(1)$ $\mathcal{O}(1)$ Action per point

$$\approx N_t^4\left(0.6 + \frac{1}{N_t^4} - 0.6\right)$$

The bigger and better the lattice, the deeper is physics buried!

Lattice gives only numbers, dimless ratios

$$\int_{\mathbf{x},\mathbf{y}} \langle \Pi^0(\tau,\mathbf{x})\Pi^0(0,\mathbf{y})\rangle = C\exp(-m_\pi\tau)$$

Pion operator

$$m_\pi a(\tau/a)$$

$$Z_\pi \bar{\psi} i\gamma_5 T^3 \psi$$

1loop asymptotic scaling:

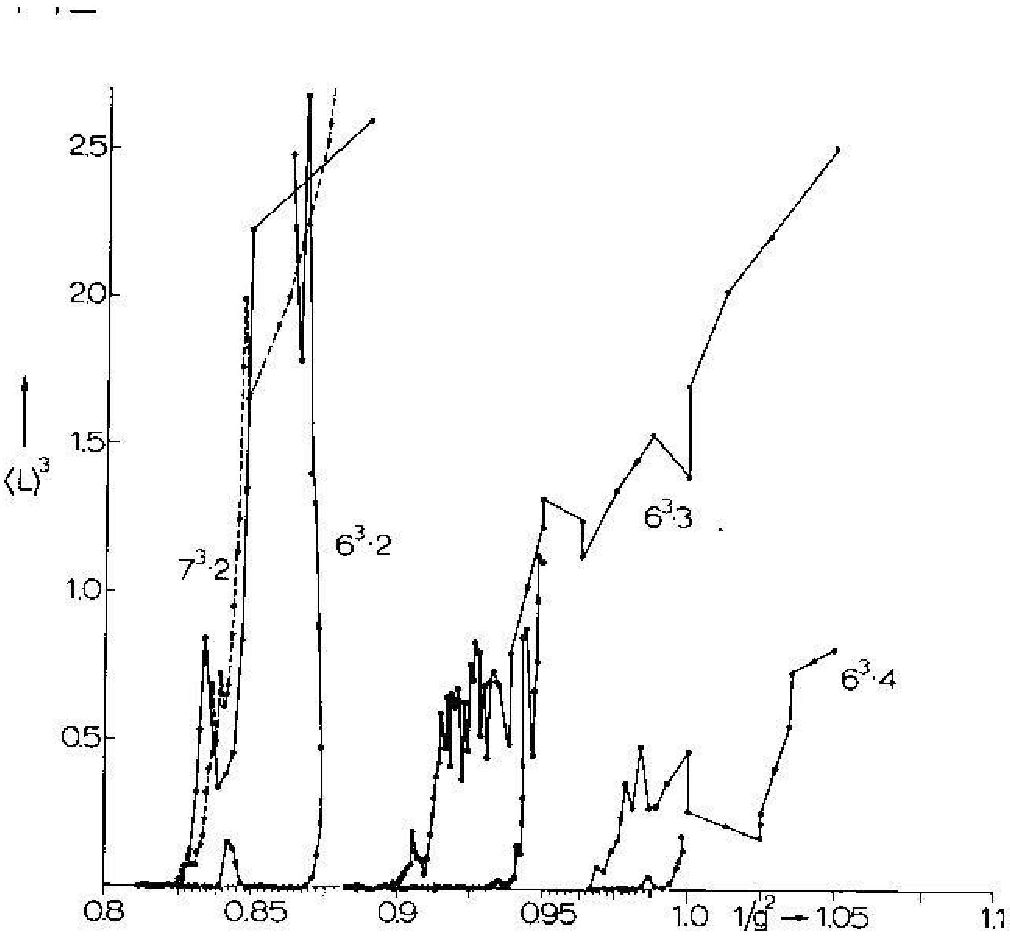
$$a = \frac{1}{\Lambda_L} e^{-\frac{1}{2\beta_0 g^2(a)}}$$

lattice Λ

bare lattice coupling

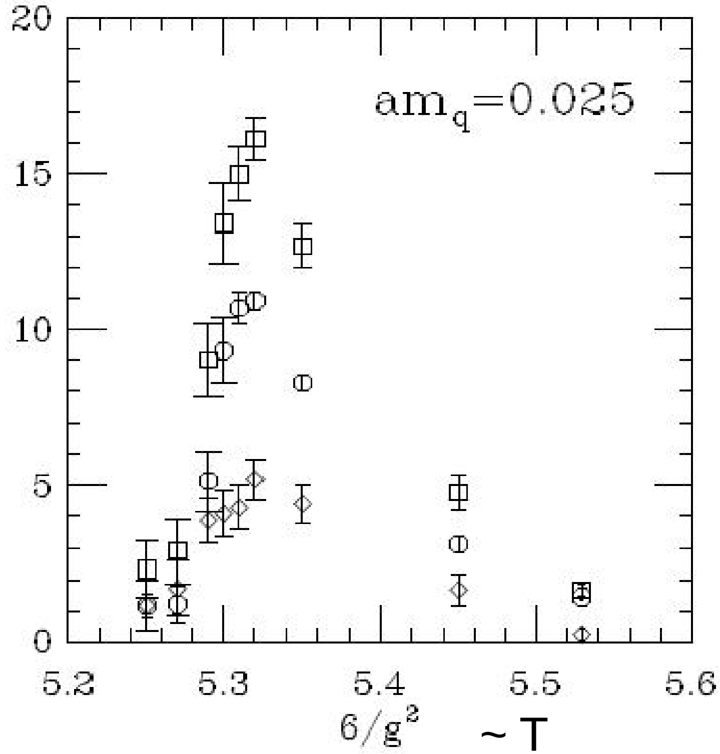
Some history:

1982, SU(3) : Kajantie-Montonen-Pietarinen



1994, $N_f = 2$: Blum- Gottlieb-Kärkkäinen-Toussaint

$$\frac{\epsilon - 3p}{T^4}$$

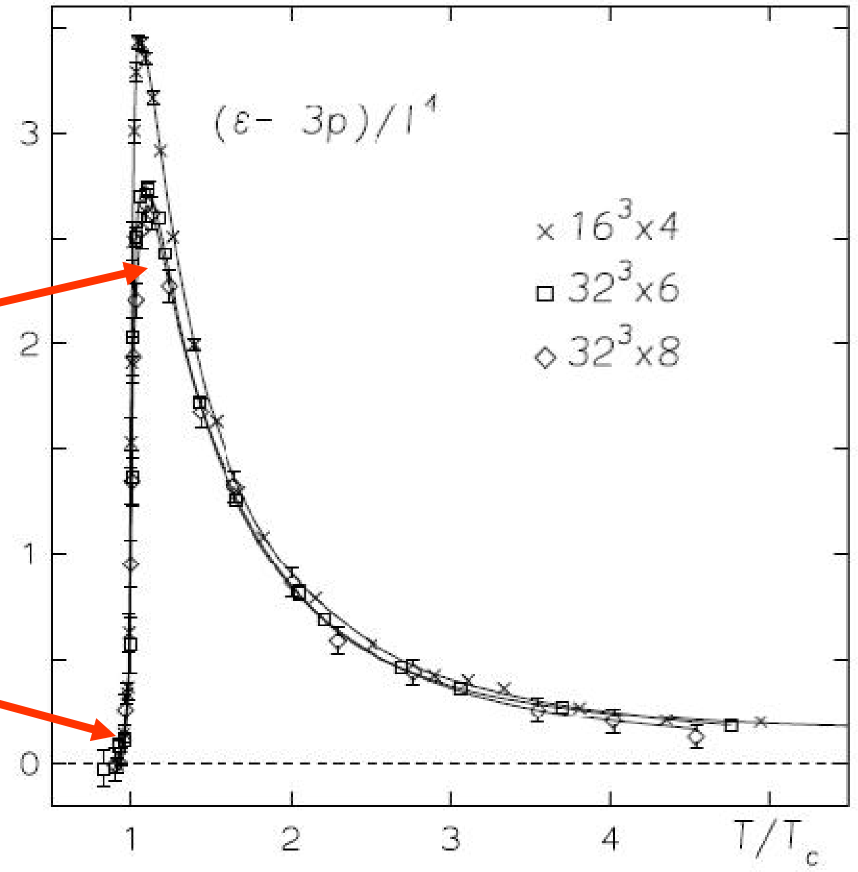


~ T

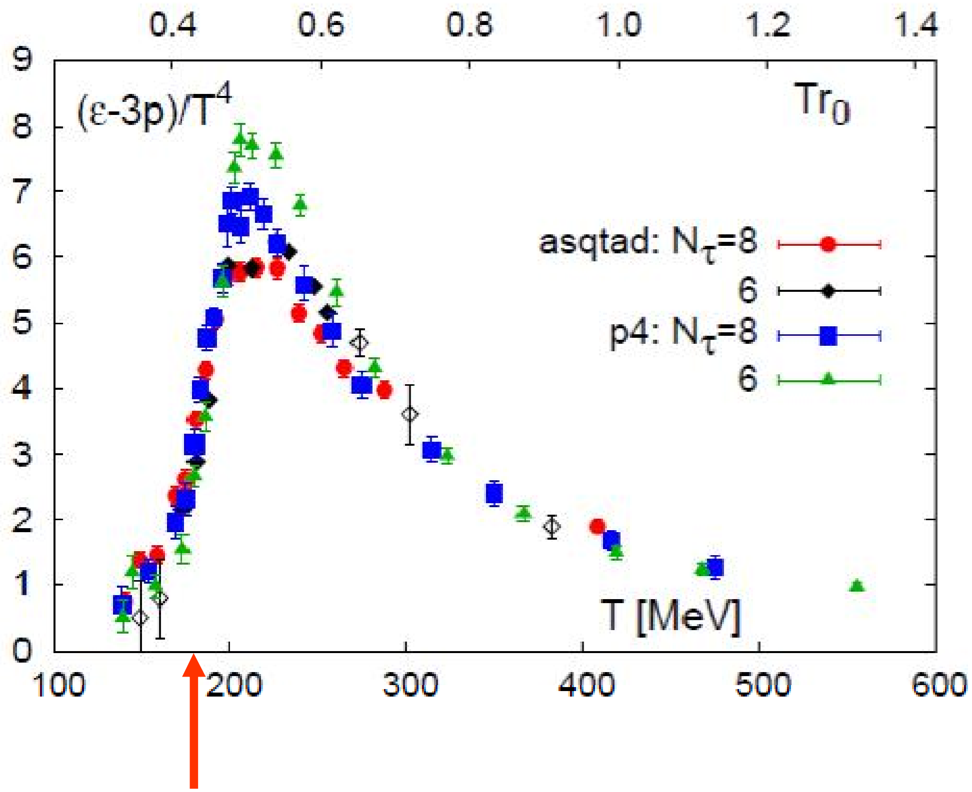
1996, pure SU(3): Boyd-Engels-Karsch-Laermann…

1st order transition

$$\frac{L}{T_c^4 N_c^2} \approx 0.3$$



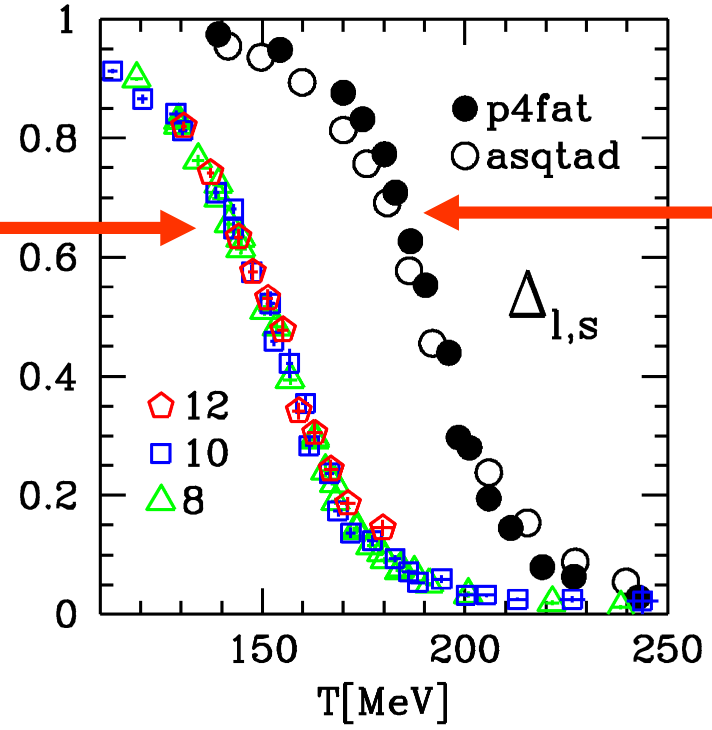
2009: N_f = 2+1 0903.4379, 23 authors



T_c

Continuous transition!!

Controversy about the value of T_c :

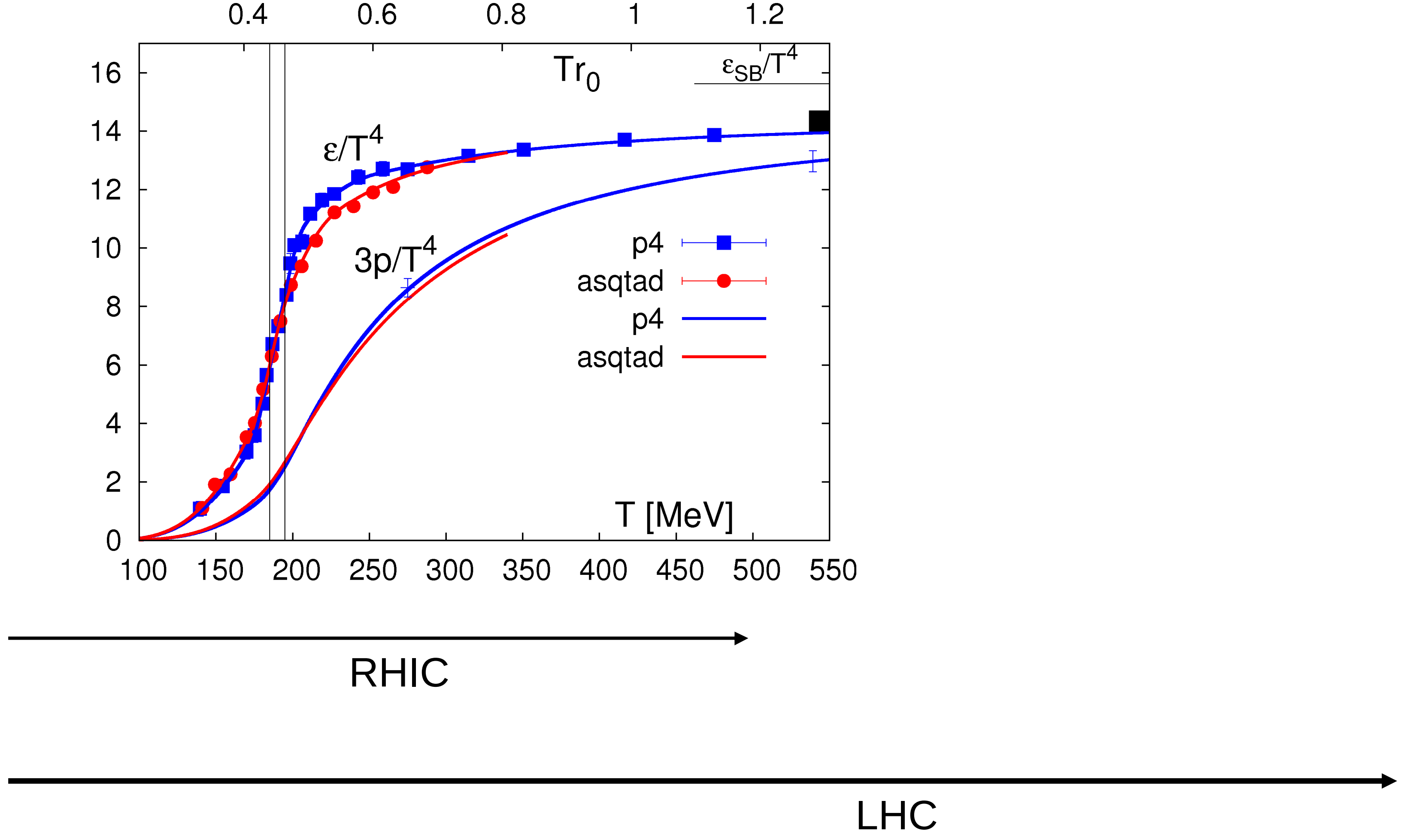


Budapest-
Wuppertal

Hot QCD,
USA-Bielefeld

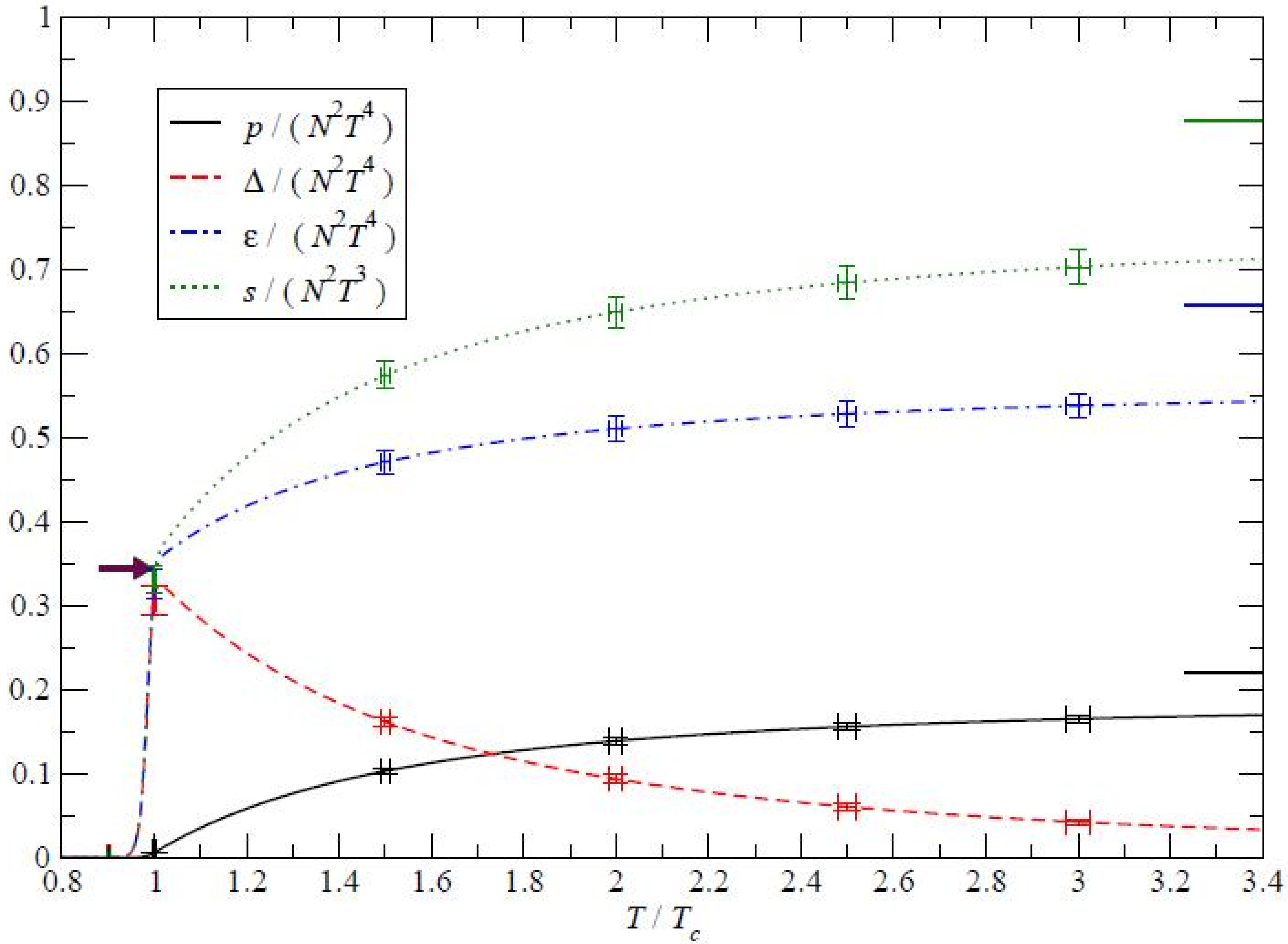
Cosmological effects?

Integrate from $\epsilon - 3p$



Extrapolation to the large-N limit

Panero 0907.3719

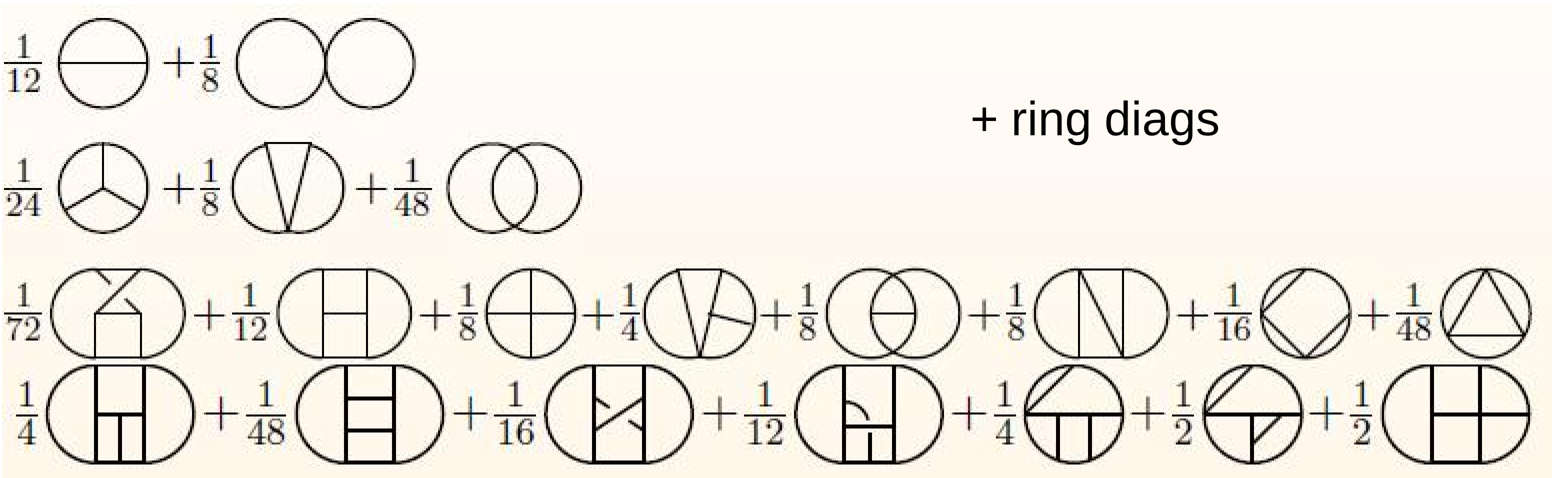


2. Perturbation theory for p(T)

$$e^{p(T)V/T} = \int \mathcal{D}A\, e^{-(\partial A + gA^2)^2}$$

$$= \int \mathcal{D}A\, e^{A\partial^2 A}\left[1 + \sum_n \frac{1}{n!}(2g\partial A \cdot A^2 + g^2 A^4)^n\right]$$

Generate vacuum diagrams:



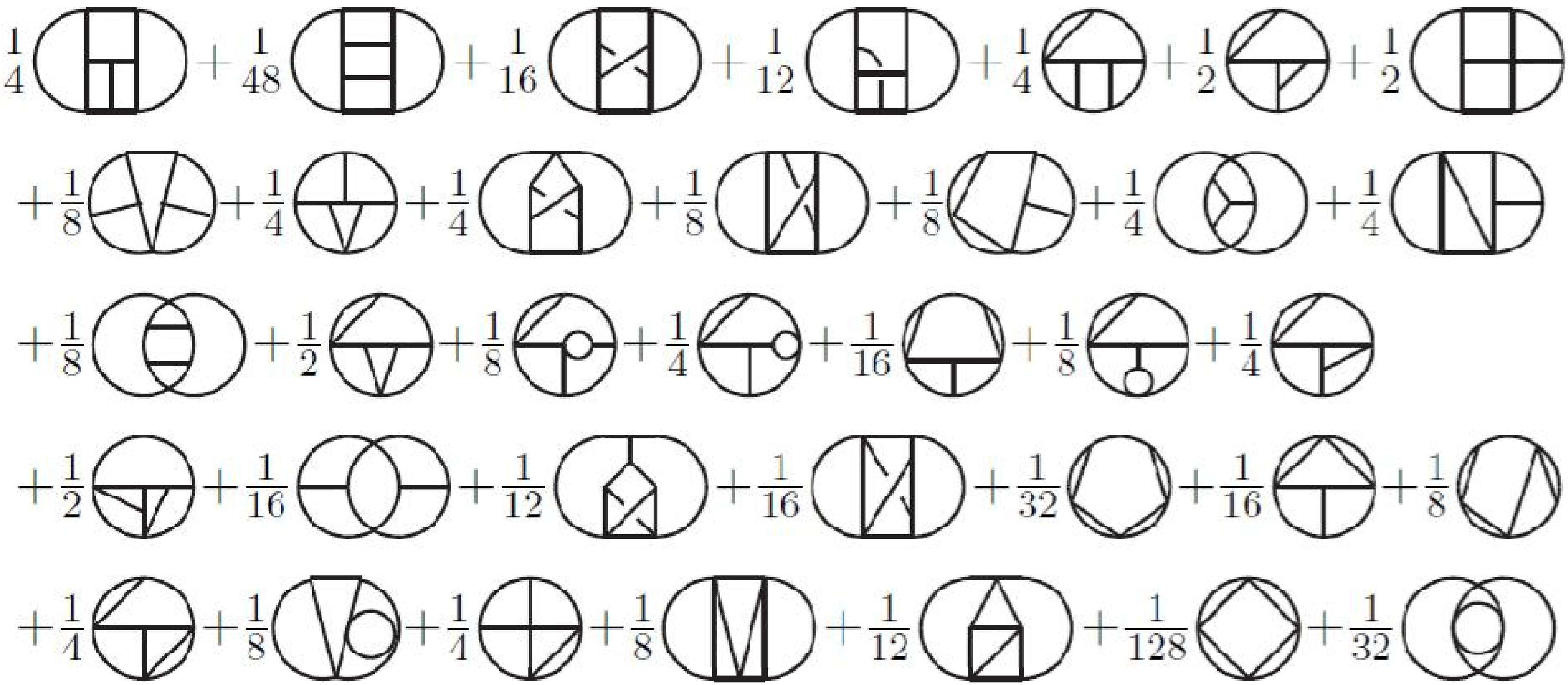
+ ring diags

$$T\sum_n \int \frac{d^{3-2\epsilon}k}{(2\pi n T)^2 + \mathbf{k}^2}$$

IR divs at k=0; physics is electric screening and magnetic sector confinement

All topologically distinct 5-loop vacuum diags;

Kajantie-Laine-Schröder
hep-ph/0109100



Exercise in futility (mathematics): generalise to n loops

No wonder QCD matter becomes strongly interacting!

Systematic way of handling IR divs: effective theories

$\frac{g^2(T)N_c}{(4\pi)^2}$

$$\boxed{\text{QCD} \equiv \text{4d YM + quarks; } |\mathbf{k}| \sim g^2T, gT, 2\pi T}$$

$\Downarrow$ perturbation theory (1)

$\frac{\sqrt{g^2(T)N_c}}{4\pi}$

$$\boxed{\text{EQCD} \equiv \text{3d YM} + A_0;\ |\mathbf{k}| \sim g^2T, gT} \quad \int d^3x \left[\tfrac{1}{4}F_{ij}^2 + (D_iA_0)^2 + m^2A_0^2 + ..\right]$$

$\Downarrow$ perturbation theory (2)

non-pert

$$\boxed{\text{MQCD} \equiv \text{3d YM; } |\mathbf{k}| \sim g^2T} \quad \int d^3x \tfrac{1}{4}F_{ij}^2$$

Get expansion of type

πT	$1 + g_{(1)}^2$		$+g_{(1)}^4 \ln$		$+g_{(1)}^6(\ln + [\text{pert}]_1) + \ldots$
E: gT		$+g_{(2)}^3$	$+g_{(2)}^4 \ln$	$+g_{(2)}^5$	$+g_{(2)}^6(\ln + [\text{pert}]_2) + \ldots$
M: g^2 T					$+g_{(3)}^6(\ln + [\text{non-pert}]) +$

All but the last term on first line is known!

IR divergences at finite T lead to an expansion of the form:

g = standard 2-loop MSbar running coupling

$$c_{\mathrm{SB}} + c_2 g^2 + c_3 g^3 + (c_4' \log g + c_4) g^4 + c_5 g^5 + (c_6' \log g + c_6) g^6 + c_7 g^7 + ..$$

c_2 Shuryak 78, c_3 Kapusta 79, c_4' Toimela 83, c_4 Arnold-Zhai 94,
c_5 Zhai-Kastening, Braaten-Nieto 95, c_6' Kajantie-Laine-Rummukainen-Schröder 03

contains logμ

$$g^2(\mu) + g^4 \left(-2\beta_0 \log \frac{T}{\mu} + c \right) + ..$$

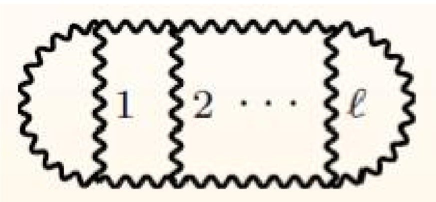
Optimize by choosing μ = const * T

Converges badly: expansion parameter g/π,
c_6 even non-perturbative

What is the meaning of ”c_6 is non-perturbative”?

From 4 loops onwards all loops contribute to order g^6 :

Linde’s dimensional argument:



$$\sim \left(T \sum_n \int d^3p \right)^{\ell+1} \frac{(gp)^{2\ell}}{[(2\pi nT)^2 + p^2 + \Pi(2\pi nT, p)]^{3\ell}}$$

(ℓ+1) loops, 2ℓ vert, 3ℓ propags

$$\sim T^{\ell+1} g^{2\ell} m^{3(\ell+1)+2\ell-6\ell} = g^6 T^4 \left(\frac{g^2 T}{m} \right)^{\ell-3}$$

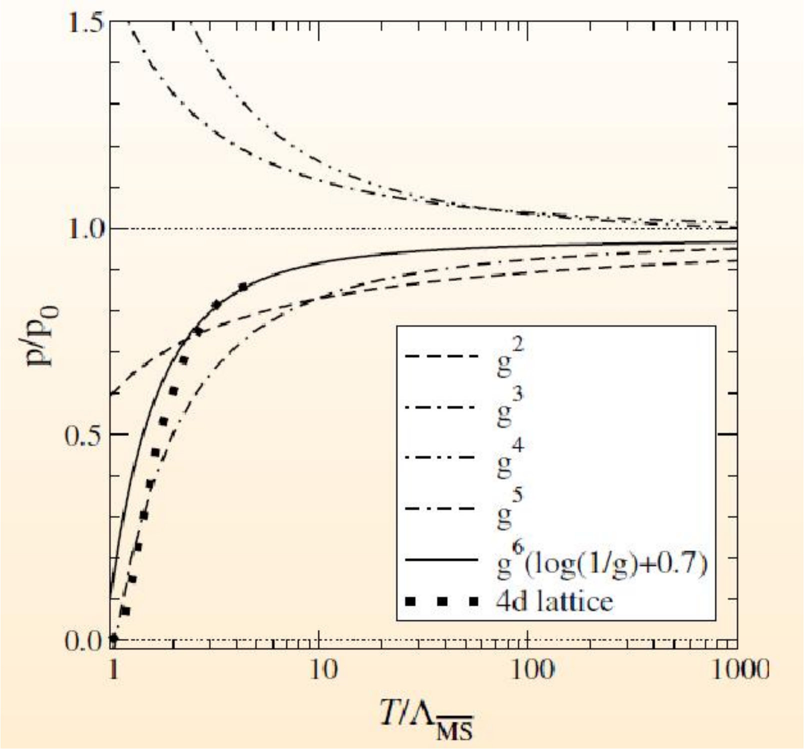
Can c_6 be numerically determined?

Yes; the nonpert lattice part is already done and matched to MSbar,

Hietanen-Kajantie-Laine-Rummukainen-Schröder-DiRenzo-Miccio-Torrero hep-lat/0412008, hep-ph/0605042

computing the 4loop sum-integrals for p(T) in strict MSbar is missing, a formidable task, some 10^8 4loop diags -> > 100 scalar master integrals!

Long ago, there was the picture of ”ideal quark-gluon gas”, but



-there is the confining magnetic sector

-pert theory converges slowly, g~2

-experiments!

a strongly coupled system

AdS/QCD

Why approach from below?
Leading corr is $1 - g^2 +..$

$$\epsilon > 3p \qquad c_s^2 < \frac{1}{3}$$

3. Beta functions

Important for later AdS applications!

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta(g) = -b_0 g^3 - b_1 g^5 - b_2 g^7 + \mathcal{O}(g^9)$$

New scheme

$$g' = G(g) = g + a_1 g^3 + a_2 g^5 + \mathcal{O}(g^7)$$

Conversely

$$g = G^{-1}(g') = g' - a_1 g'^3 + (3a_1^2 - a_2) g'^5 + \mathcal{O}(g'^7)$$

$$\Rightarrow \beta(g') = \mu \frac{\partial G(g(\mu))}{\partial \mu} = \beta(g) G'(g)|_{g=G^{-1}(g')}$$

$$= -b_0 g'^3 - b_1 g'^5 - (b_2 - 3a_1^2 b_0 + 2a_2 b_0 - 2a_1 b_1) g'^7 + \dots$$

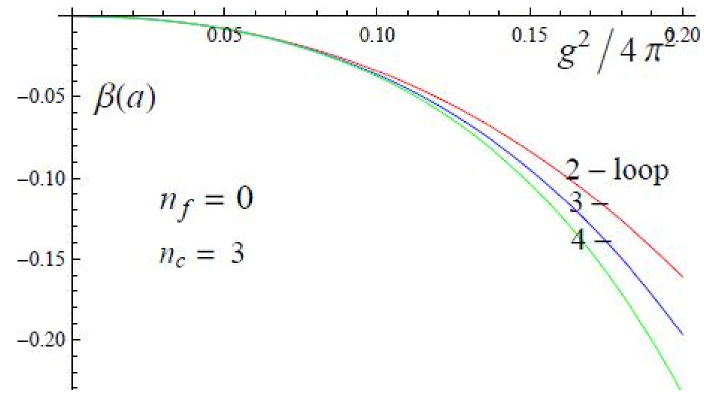
If $\beta(g_*) = 0$ then also $\beta(g'_*) = 0$

IR fixed pt, b_0, b_1 are scheme independent

QCD:

$$\frac{da}{d\ln\mu^2} = \beta(a) \qquad a = \alpha_s/\pi = g^2/4\pi^2$$

$$= -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 - \beta_3 a^5 + O(a^6)$$



$$\beta_0 = \frac{1}{4}\left[11 - \frac{2}{3}n_f\right]$$

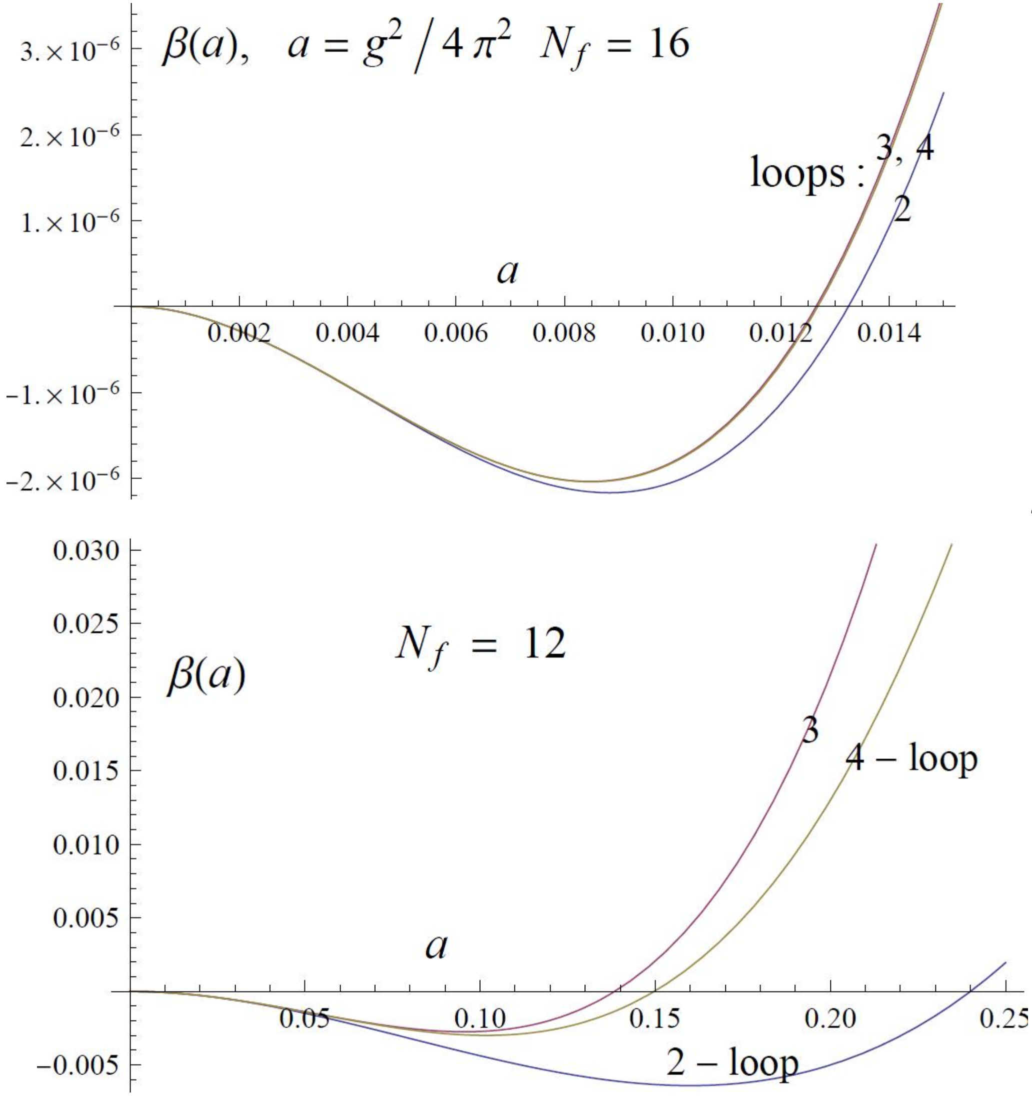
$$\beta_1 = \frac{1}{16}\left[102 - \frac{38}{3}n_f\right]$$

$$\beta_2 = \frac{1}{64}\left[\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right]$$

$$\beta_3 = \frac{1}{256}\left[\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3\right]$$

Vermaseren-Larin-van Ritbergen hep-ph/9703284, 9701390

Searching for IRFP:



Reliable, a << 1, Banks-Zaks

$$\beta(a) = \gamma(a - a_*) + ..$$

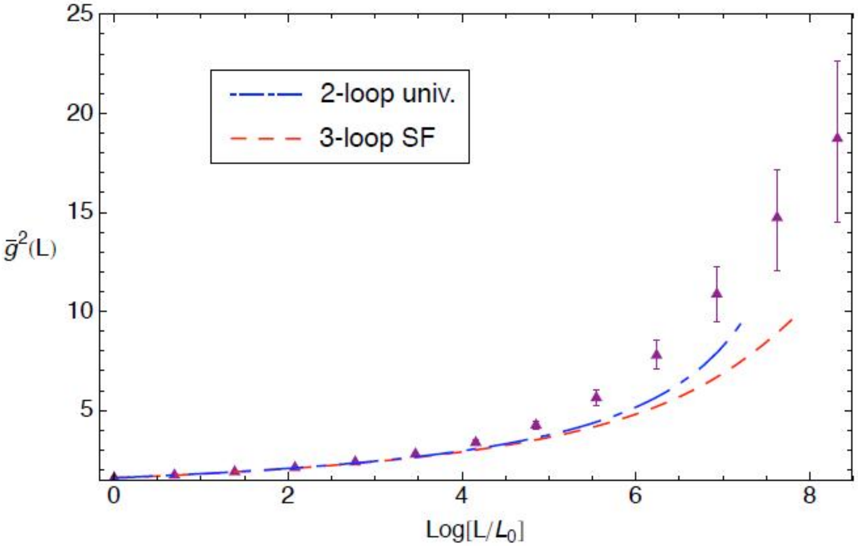
$$a = a_* + C\mu^{2\gamma} + ..$$

γ depends on scheme!

Semireliable, a < 1

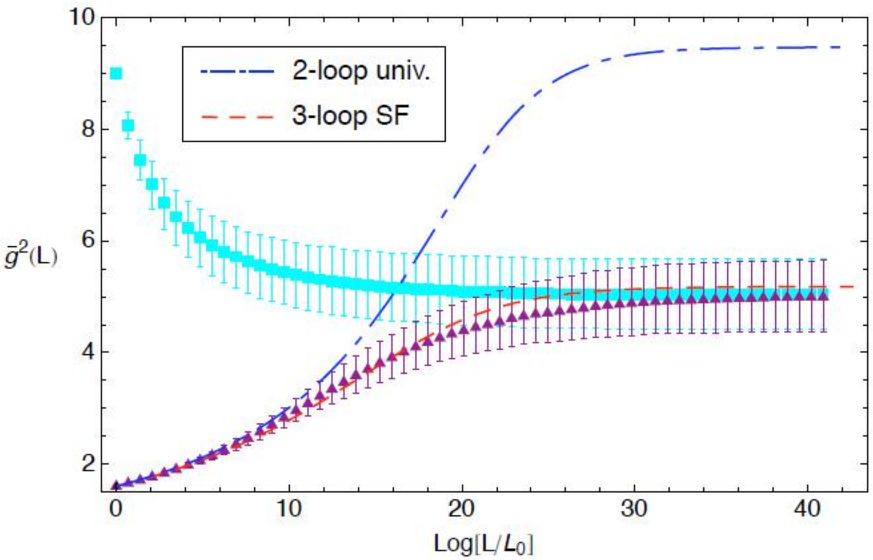
Lattice studies using SF coupling

Appelquist-Fleming-Neil 0901.3766



$N_f = 8$ coupling diverges

g(L), L=lattice size



$N_f = 12$ coupling driven to IRFP at $\approx$ pert value

Contested by

Fodor-Holland-Kuti-Nogradi-Schroeder 0911.2463

highly non-trivial to conclude!

Search for theories with IRFP is a hot topic in lattice field theory

Catterall-Giedt-Sannino-Schneible 0807.0792
Hietanen-Rummukainen-Tuominen 0904.0864
Del Debbio-Lucini-Patella-Pica-Rago 0907.3896

QCD: anomalous dimension $$\mu\frac{\partial m(\mu)}{\partial\mu} = \gamma(g)m(\mu)$$

$$\begin{aligned}\frac{d\ln m_q}{d\ln\mu^2} &= \gamma_m(a) \qquad\qquad a = \alpha_s/\pi = g^2/4\pi^2 \\ &= -\gamma_0 a - \gamma_1 a^2 - \gamma_2 a^3 - \gamma_3 a^4 + O(a^5)\end{aligned}$$

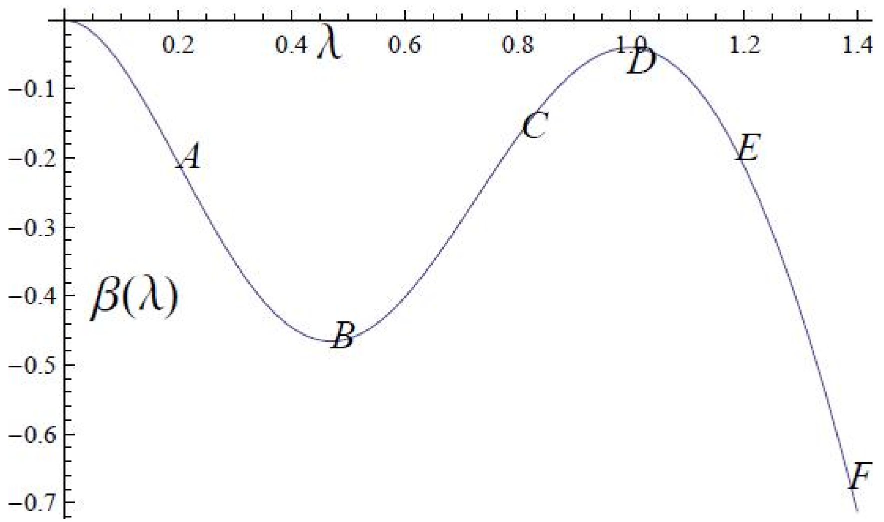
$$\begin{aligned}\gamma_0 &= 1 \\ \gamma_1 &= \frac{1}{16}\left[\frac{202}{3} - \frac{20}{9}n_f\right] \\ \gamma_2 &= \frac{1}{64}\left[1249 + \left(-\frac{2216}{27} - \frac{160}{3}\zeta_3\right)n_f - \frac{140}{81}n_f^2\right] \\ \gamma_3 &= \frac{1}{256}\left[\frac{4603055}{162} + \frac{135680}{27}\zeta_3 - 8800\zeta_5 + \left(-\frac{91723}{27} - \frac{34192}{9}\zeta_3 + 880\zeta_4 + \frac{18400}{9}\zeta_5\right)n_f \right.\\ &\left. + \left(\frac{5242}{243} + \frac{800}{9}\zeta_3 - \frac{160}{3}\zeta_4\right)n_f^2 + \left(-\frac{332}{243} + \frac{64}{27}\zeta_3\right)n_f^3\right] \qquad (15)\end{aligned}$$

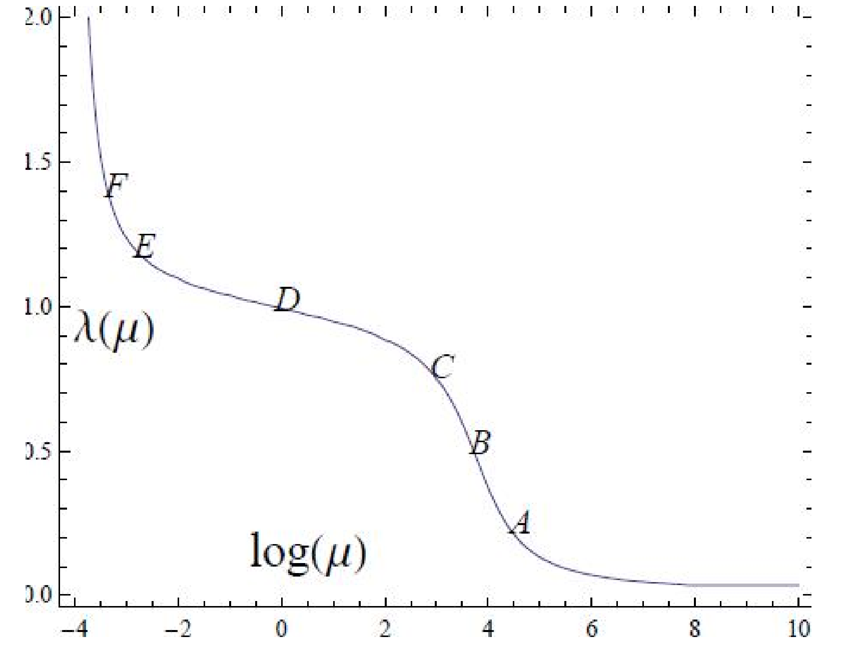
$$m' = mF(g) \Rightarrow \mu\partial_\mu m' = \gamma mF + m\beta(g)F'(g) \qquad \gamma' = \gamma + \beta(g)F'(g)/F$$

γ scheme independent at IRFP.

Walking coupling (technicolor models)

$$\beta(\lambda) = -c\lambda^2 \frac{(1-\lambda)^2 + e}{1 + a\lambda^3} \qquad c = 8,\ \ a = 1,\ \ e = 0.01$$





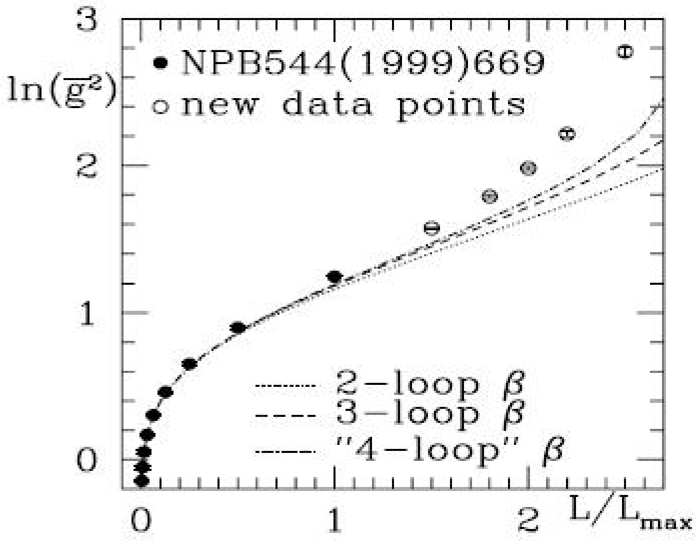
Coupling runs -> condensate walks
Coupling walks -> condensate runs (want this)

Schrödinger functional coupling

1st principle lattice method $Z \sim e^{\frac{1}{g^2(L)} E^2(\eta)}$ $\frac{1}{g^2(L)} \sim \frac{\partial \log Z}{\partial \eta}$

Impose a constant color electric field, dial with η on bdry

Scale = lattice size L



Coupling grows exponentially in the IR

$$g^2(L) \sim e^{mL}$$

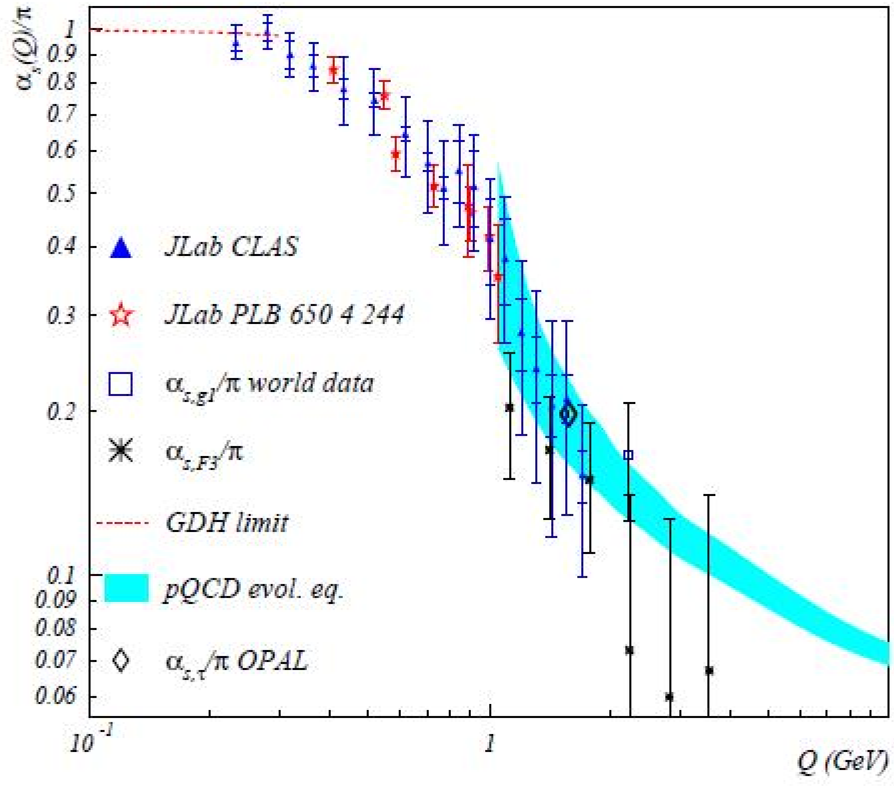
$$m \sim m_{\text{glueball}}$$

$$\beta(\lambda) = -\lambda\left(\log\lambda + \mathcal{O}(1)\right)$$

Lüscher et al hep-lat/9207009,
Lüscher hep-lat/9802029
Heitger-Simma-Sommer-Wolff, hep-lat/0110201

Freezing coupling

Phenomenologists want the coupling to freeze in the IR:



Coupling const in IR
Massive states!

0803.4119

SuSy beta function

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{\vartheta}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} + \frac{i}{g^2} \lambda^{a\alpha} \mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{a\dot{\beta}}$$

$$= \frac{1}{4} \left(\frac{1}{g^2} - i\,\frac{\vartheta}{8\pi^2} \right) \int d^2\theta \, \mathrm{Tr}\, W^2 + \mathrm{H.c.}\,,$$

Exact gluino condensate, SU(2):

$$\langle \mathrm{Tr} \lambda\lambda \rangle = \pm \frac{2^5 \pi^2}{\sqrt{5}} M_{\mathrm{PV}}^3 \frac{1}{g^2} \exp\left\{ -\frac{4\pi^2}{g^2} \right\}$$

Matter superfields:

$$3 - \frac{N_f}{N_c}(1-\gamma)$$

$$M \frac{\partial \langle \lambda\lambda \rangle}{\partial M} = 0 \Rightarrow \beta(g^2) = -\frac{3g^4}{8\pi^2/N_c - g^2}$$

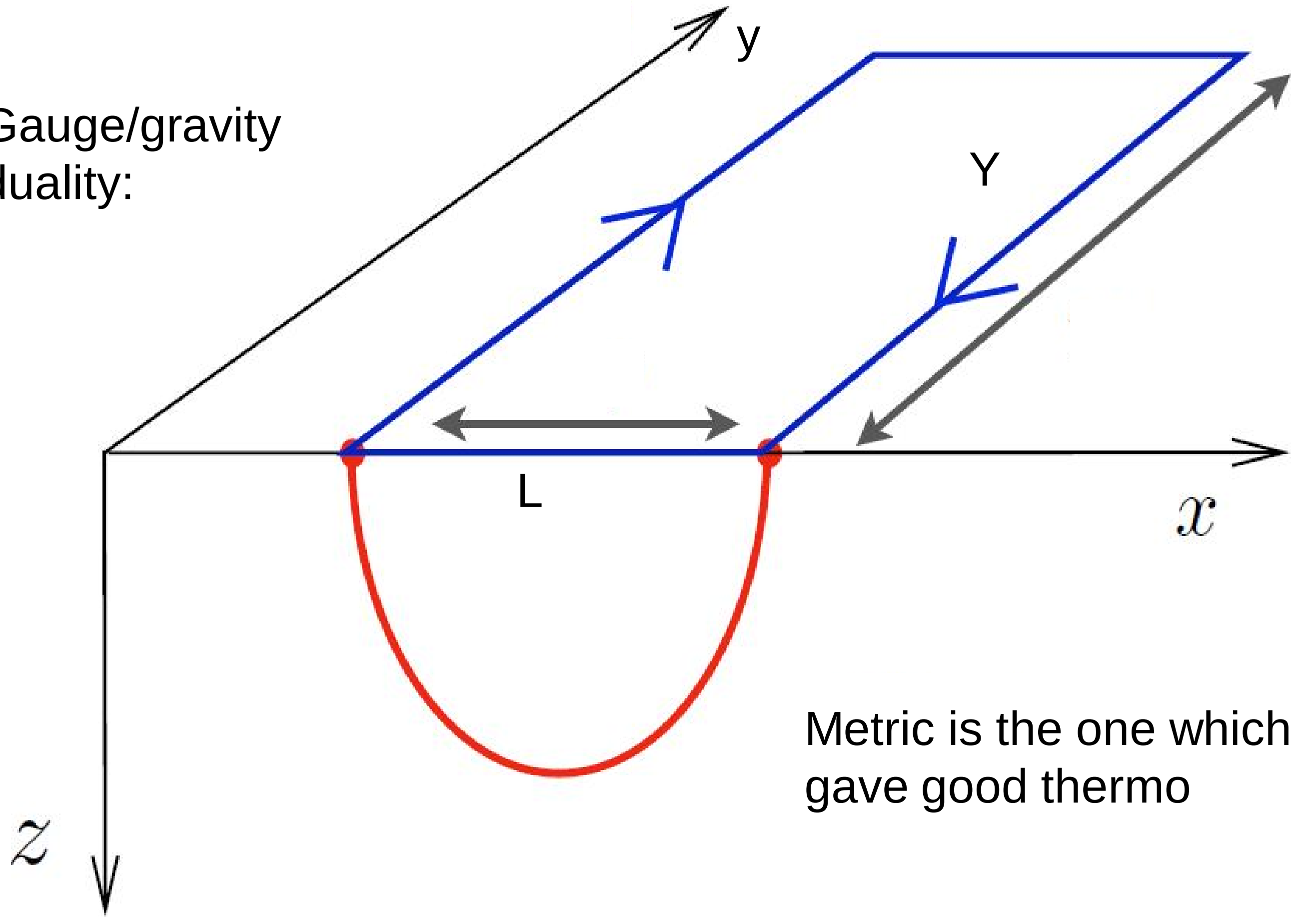
Novikov-Shifman-Vainshtein-Zakharov, Shifman hep-th/9906049

4. Spatial string tension σ(T): lattice and pert theory

Measure <Wilson loop> for x,y loop in a finite T lattice, $0 < \tau < 1/T \quad N_t \ll N_s$

Large loop = $\exp[-\sigma_s XY]$

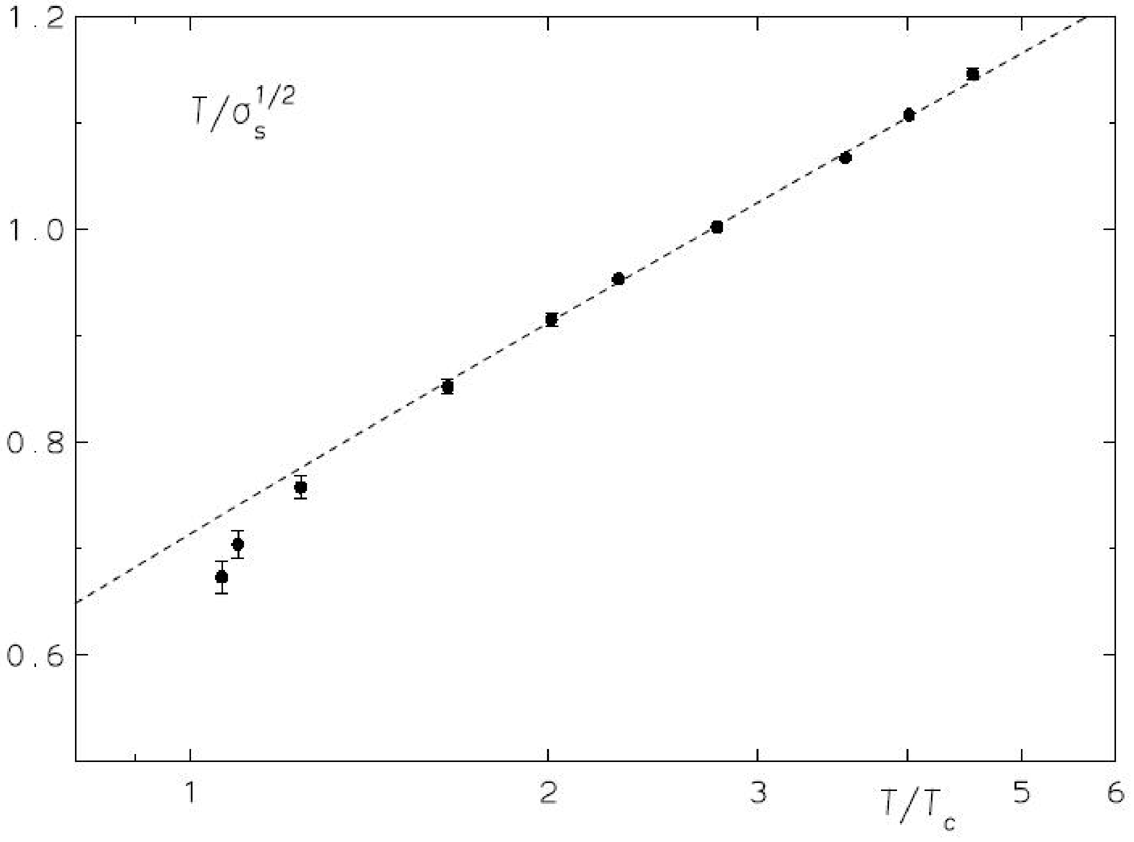
Gauge/gravity duality:



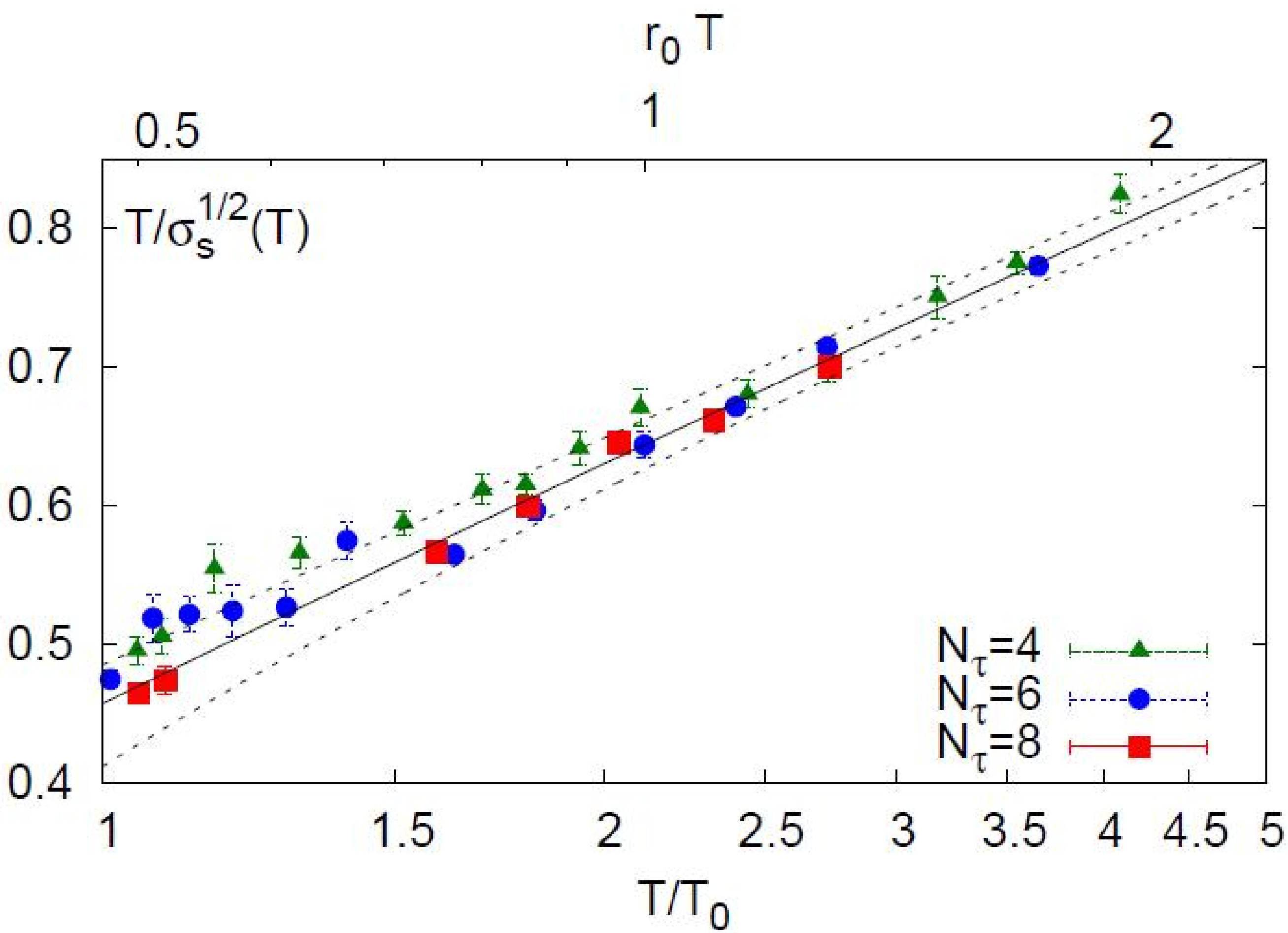
Metric is the one which gave good thermo

Values for SU(3) $8 \cdot 32^3$

Boyd et al, hep-lat/9602007



Nf = 2 + 1 $N_t = 4, 6, 8 \quad N_s \geq 4N_t$ Cheng et at 0806.3264



But can also measure σ in the 3d spatial sector without any 4th dim, string tension in 3d SU(3) Yang-Mills

Alanen-Kajantie-SuurUski 0905.2032, PRD

$$\sqrt{\sigma_s} = 0.553(1) g_M^2$$

non-pert number — the only dimful quantity in 3d Y-M

Teper 0812.0085

What do you predict for $\sigma_s(T)$?

pert

For dimensional reasons $g_M^2 = g^2(T)T$

Expect and get sth like

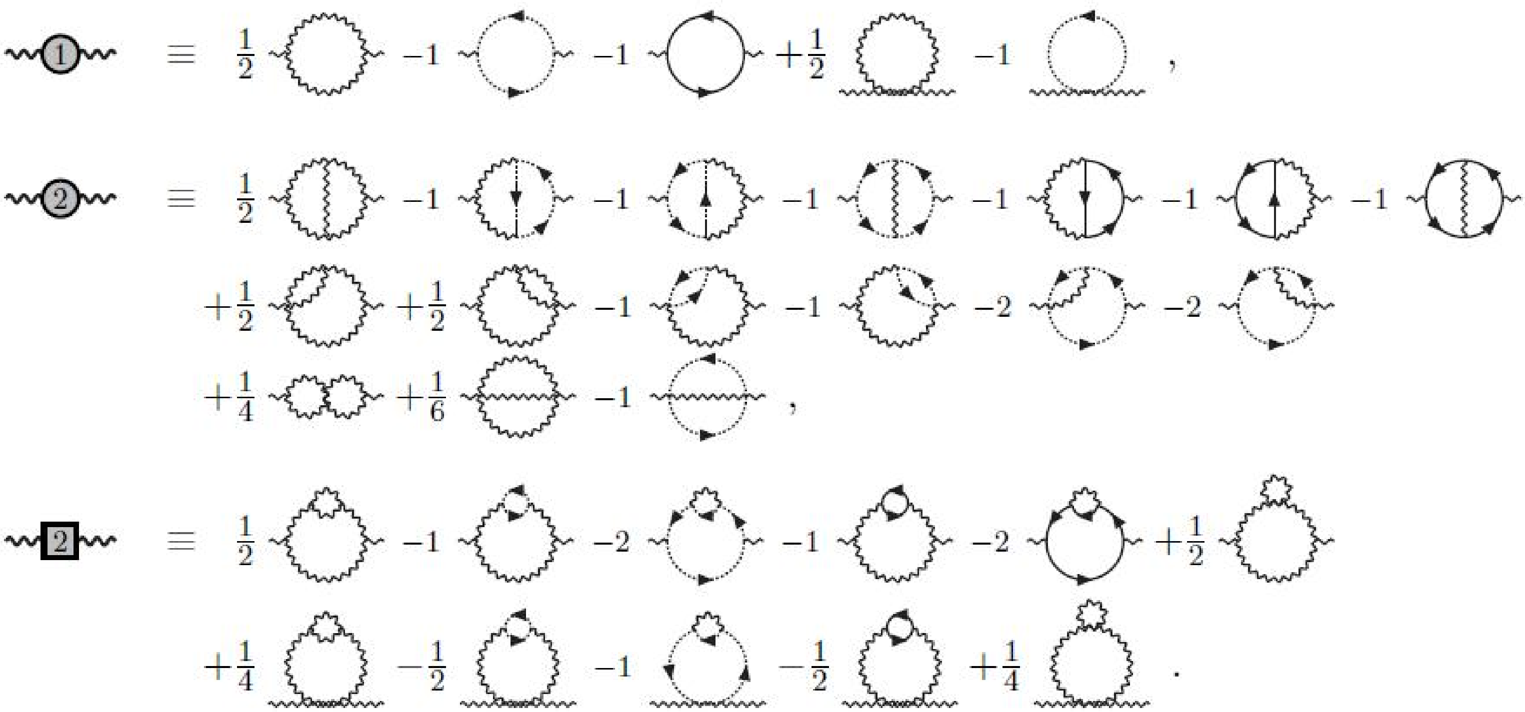
$$\frac{T}{\sqrt{\sigma_s}} = 1.81 \frac{1}{g^2(T)} = 0.25 \left[\log \frac{T}{\Lambda_\sigma} + \frac{51}{121} \log\left(2 \log \frac{T}{\Lambda_\sigma} \right) \right]$$

Can fit from data → Λ_σ = ? $= T_c/7.753$

Pert theory is **computing** this Laine Schröder hep-ph/0503061

Typical procedure:

Identify first the diags to be computed:



Evaluate them by symbolic computation, expand to 2nd order in momentum

Get

$$g_{\rm E}^2/T = g^2(\bar\mu) + \frac{g^4(\bar\mu)}{(4\pi)^2}\Big[-\beta_0 \ln\Big(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\Big) + \frac{1}{3}N_c\Big]$$
$$+ \frac{g^6(\bar\mu)}{(4\pi)^4}\Big\{-\beta_1 \ln\Big(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\Big) + \Big[\beta_0 \ln\Big(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\Big) - \frac{1}{3}N_c\Big]^2$$
$$- \frac{1}{18}N_c^2\Big[-341 + 20\zeta(3)\Big]\Big\} + \mathcal{O}(g^8)$$

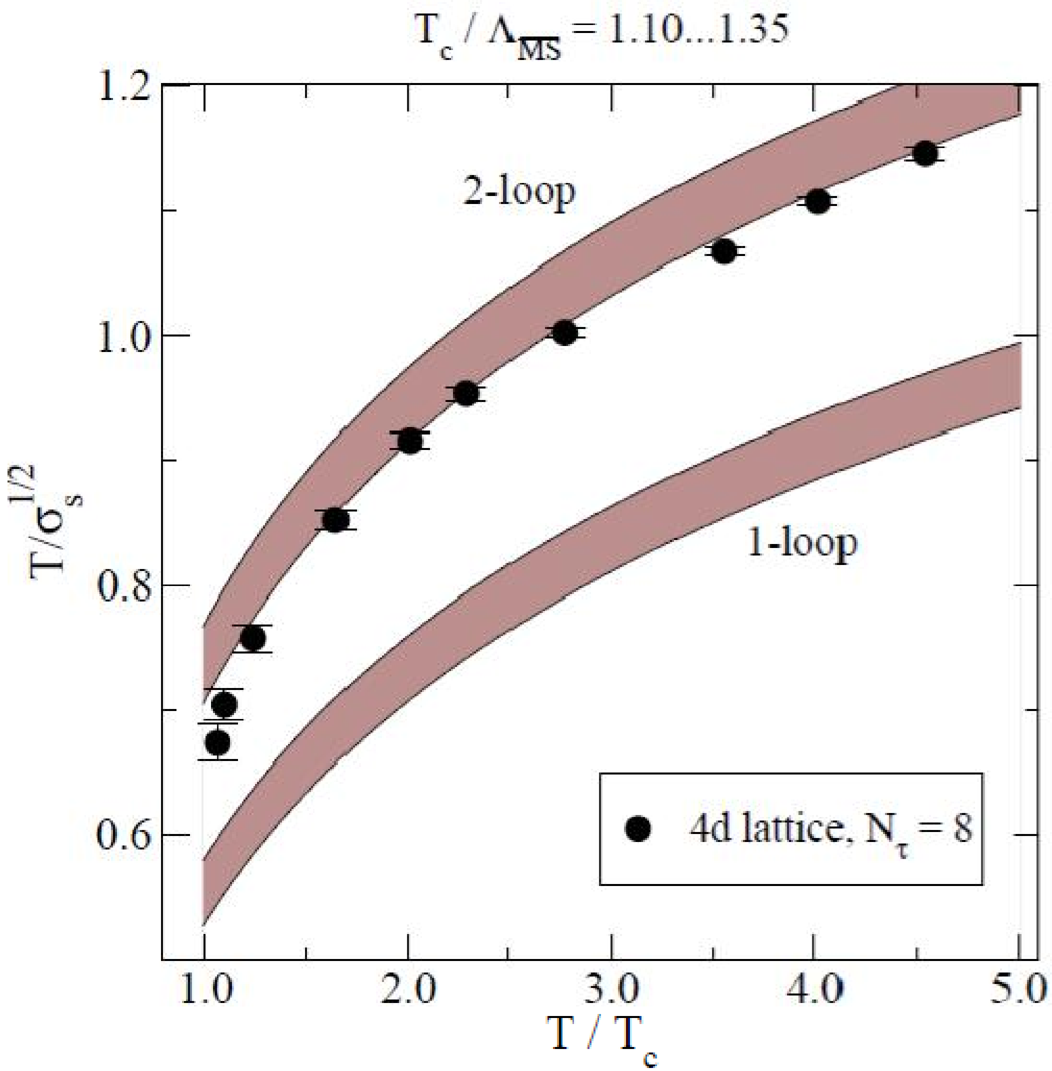
and optimize the scale here so that, for example, the g^4 term vanishes:

$$\bar\mu_{\rm op} = 4\pi e^{-\gamma_{\rm E} - 1/22} T = 6.742\, T = \frac{7.753 T}{T_c} \quad \text{since} \quad T_c = 1.15\Lambda_{\overline{\rm MS}}$$

So the scale in the MSbar coupling has been evaluated!

$$= g^2(\bar\mu_{\rm op}) + \frac{g^6(\bar\mu_{\rm op})}{(4\pi)^4}\frac{1}{198}N_c^2[3547 - 220\zeta(3)]$$

Varying the scale factor gives Laine-Schröder hep-ph/0503061



Reproducible
well defined

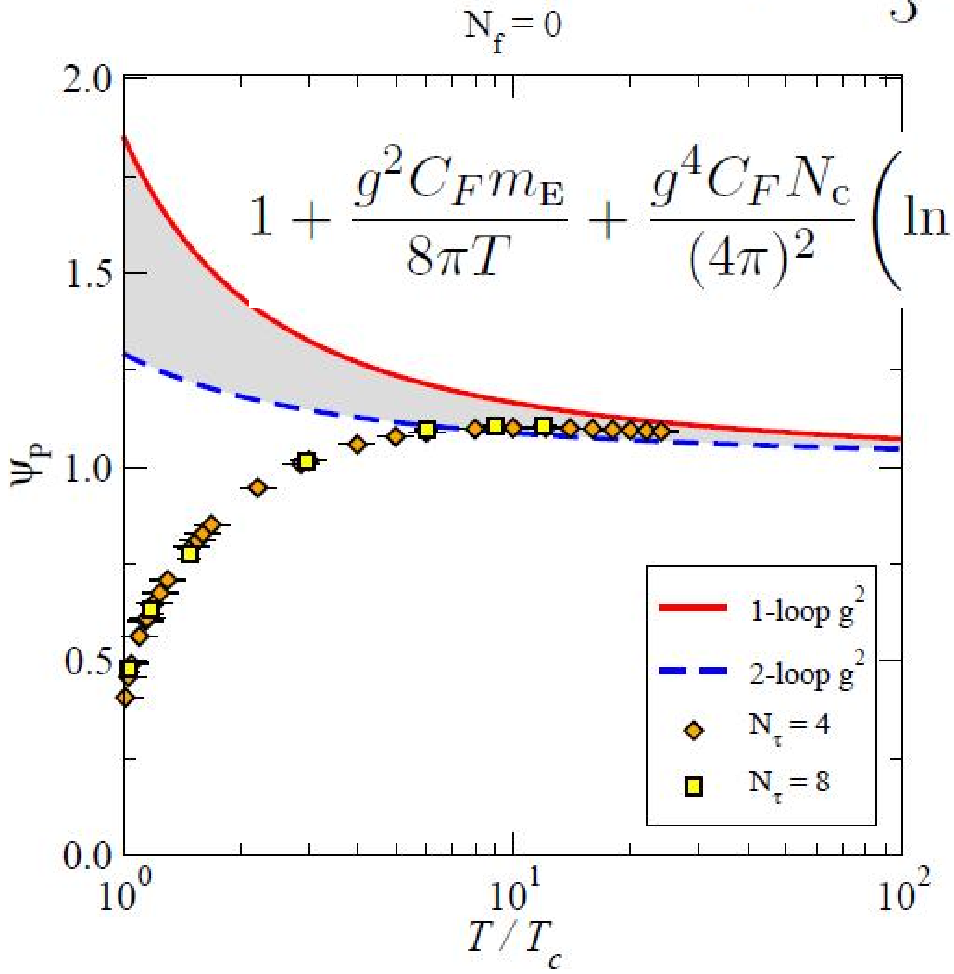
3 loop ?
Lattice cont ?

What does AdS/QCD give?

Polyakov line

$$m_E^2 = \frac{N_c}{3} g^2 T^2$$

Non-local operator!!



$$1 + \frac{g^2 C_F m_\mathrm{E}}{8\pi T} + \frac{g^4 C_F N_\mathrm{c}}{(4\pi)^2}\left(\ln\frac{m_\mathrm{E}}{2T} + \frac{3}{4}\right)$$

Burnier-Laine-Vepsalainen 0911.3480

Gravity duals of finite T QCD

5. Gravity+scalar/hot QCD

- add 5th dimension z > 0, z=0 is boundary

- write down Einstein gravity for a metric+scalar ansatz:

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R - \tfrac{4}{3} (\partial_\mu \phi)^2 + V(\phi)\right] \quad V(0) = \frac{12}{\mathcal{L}^2}$$

$$ds^2 = b^2(z) \left[-f(z)dt^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)}\right] \qquad \lambda(z) = e^{\phi(z)}$$

flat BH

- find solutions which are "asymptotically (z-> 0) AdS" $\sim N_c g^2$

$$ds^2 = \frac{\mathcal{L}^2}{z^2} \left[-dt^2 + d\mathbf{x}^2 + dz^2\right] \qquad \lambda(0) = 0 \quad \phi(0) = -\infty$$

Gürsoy-Kiritsis-Mazzanti-Nitti 0903.2859

Alanen-Kajantie-SuurUski 0905.2032, 0911.2114, 0912.4128

Galow-Megias-Nian-Pirner 0911.0627

Noronha 0910.1261, 1001.3155

Järvinen-Sannino 0911.2462

Panero 0912.2448

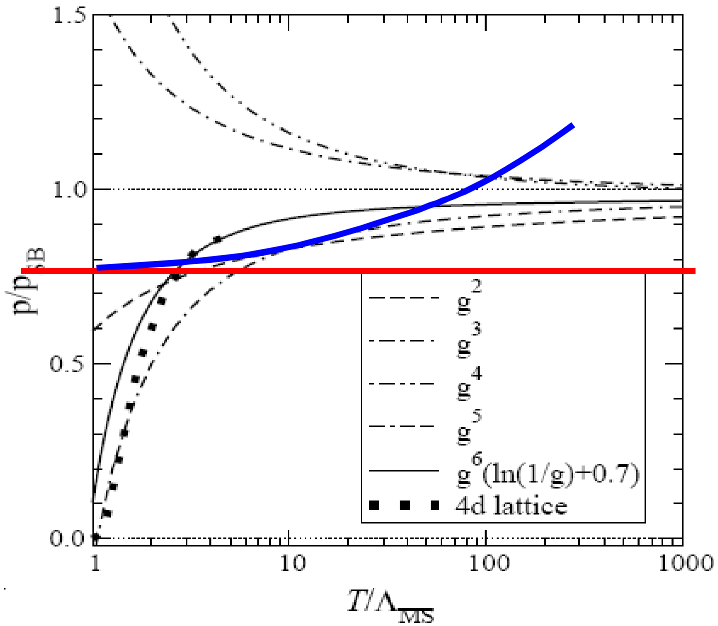
5.1 No scalar baseline: conformally invariant solution: $p = aT^4$

$$b(z) = \frac{\mathcal{L}}{z} \qquad f(z) = 1 - \frac{z^4}{z_h^4} \qquad \lambda = 0 \qquad \beta(\lambda) = 0$$

$$\pi T = \frac{1}{z_h} \qquad S = \frac{A}{4G_5} = \frac{1}{4G_5}\frac{\mathcal{L}^3}{z_h^3} \cdot V_3 = \frac{\pi^2 N_c^2}{2} T^3 \cdot V_3$$

String theory: $$\frac{\mathcal{L}^3}{G_5} = \frac{2N_c^2}{\pi}$$

The famous ¾:



5.2 No scalar: AdS BH with curved horizon. $p/T^4 \sim 1 - 1/(T\mathcal{L})^2$

Could there be effects of scale invariance breaking by spatial curvature?

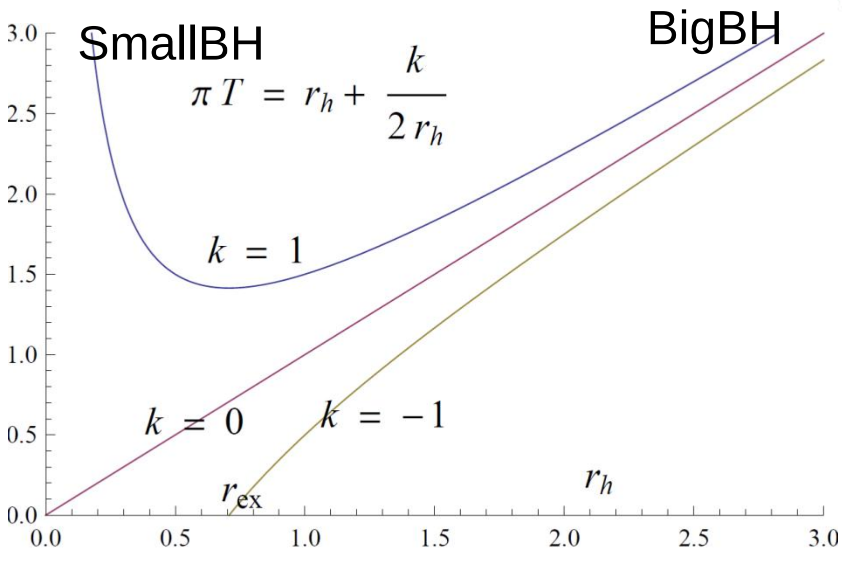
$$S = \frac{1}{16\pi G_{d+1}} \int_M d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{\mathcal{L}^2} \right)$$

Gibbons-Wiltshire NPB 1987
Witten 1998

$$ds^2 = -F dt^2 + \frac{dr^2}{F} + r^2 d\Omega_{d-1}^2(k),$$

$$F(\hat{r}) = \hat{r}^2 + k - \frac{\mu}{\hat{r}^{d-2}}, \qquad \hat{r} = r/\mathcal{L} \qquad F(r_h) = 0$$

$d = 4, \mathcal{L} = 1$



$$\pi T = r_h + \frac{k}{2r_h} \qquad s = \frac{r_h^3}{4G_5}$$

$$M \sim \mu, \qquad d\epsilon = T ds$$

EoS = ?

Method1: Integrate s=p'(T), p = -f

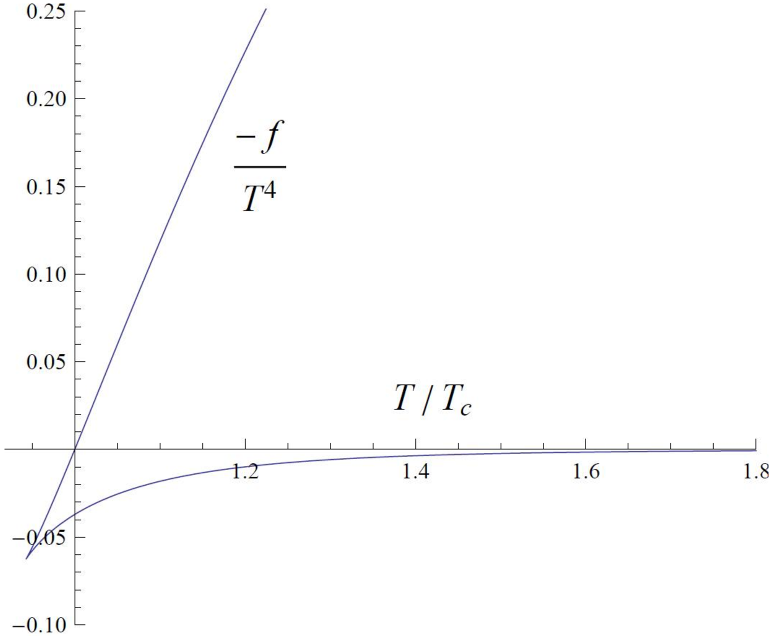
$$16\pi G_5 \mathcal{L} = 1$$

$$T = r_h + \frac{1}{2r_h} \qquad -f = \int_0^{r_h} dr_h \frac{dT}{dr_h} s - f(r_h = 0)$$

Note start from r=0, large T!!

vacuum
no BH

$$-f + f_{\text{vac}} = r_h^4 - r_h^2$$



Phase transition at $f=f_{vac}$
$r_h = 1$, T=3/2

T > T_c BH phase dominates
Hawking-Page
T < T_c $r_h = 0$ phase dominates

Method2: Holographic $T_\mu^{\ \nu}$

Kajantie-Louko-Tahkokallio 0809.4875

Coord trafo r -> z to Fefferman-Graham form:

$$\mu = \frac{4}{\hat{z}_+^4} - \frac{k^2}{4}$$

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{\left(1-\frac{\hat{z}^4}{\hat{z}_+^4}\right)^2}{\left(1-\frac{k\hat{z}^2}{2}+\frac{\hat{z}^4}{\hat{z}_+^4}\right)}dt^2 + \left(1-\frac{k\hat{z}^2}{2}+\frac{\hat{z}^4}{\hat{z}_+^4}\right)\mathcal{L}^2\, d\Omega_3^2(k) + dz^2\right]$$

Thus boundary z=0 metric is static FRW and

Skenderis

$$T_\mu^{\ \nu} = (r_h^4 + kr_h^2 + \tfrac{1}{4}\,k^2)\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \epsilon = 3p$$

Note: non-zero vacuum is explicitly evaluated

Method3: Regularised gravity action

$$Z = \exp[-f\frac{V}{T}] = \exp[-S_{\rm grav}]$$

$$S = \frac{1}{16\pi G_5}\left\{\int d^5x\sqrt{-g}\left(\frac{8}{\mathcal{L}^2}\right) - \int d^4x\sqrt{-\gamma}\left[2K - \frac{6}{\mathcal{L}} + \frac{\mathcal{L}}{2}R(\gamma) + (\log\ \text{term})\right]_{\hat{z}=\hat{\epsilon}}\right\}$$

Balasubramanian-Kraus

$$-f = r_h^4 - kr_h^2 - \frac{3}{4}k^2 \qquad f_{\rm vac} = \frac{3}{4}k^2 \quad (k>0)$$

$$p + f = T_1^{\ 1} + f = 2kr_h^2 + k^2$$

$$f = \epsilon - Ts \neq -p$$

Find failure of canonical ensemble (can use microcanonical!)

Bottom line: Two conformal phases in curved space

6. IHQCD: Gravity + scalar

Need three eqs for b(z), ϕ(z), f(z) $ds^2 = b^2(z)\left[-f(z)dt^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)}\right]$ $\lambda(z) = e^{\phi(z)}$

$$6\frac{\dot{b}^2}{b^2} + 3\frac{\ddot{b}}{b} + 3\frac{\dot{b}}{b}\frac{\dot{f}}{f} = \frac{b^2}{f}V(\phi)$$

AKS

$$6\frac{\dot{b}^2}{b^2} - 3\frac{\ddot{b}}{b} = \frac{4}{3}\dot{\phi}^2,$$

$$\frac{\ddot{f}}{\dot{f}} + 3\frac{\dot{b}}{b} = 0,$$

$$\beta(\lambda) = b\frac{d\lambda}{db}$$

GKMN

Start from $V(\phi) =$

$$\frac{12}{\mathcal{L}^2}\left\{1 + V_0\lambda + V_1\lambda^{4/3}[\log(1 + V_3\lambda^2)]^{1/2}\right\}$$

$$1 + V_0\lambda + V_1\sqrt{V_3}\lambda^{7/3} + \mathcal{O}(\lambda^{13/3})$$

λ runs with $b(z) \sim \mathcal{L}/z$ as energy scale

1. Beta function approach: start from the beta function of bdry field theory; derive V(ϕ) from the 1st equation!
2. Potential approach: start from a given V(ϕ) and derive beta function from λ,b

Key equation $$\beta(\lambda) = b\frac{d\lambda}{db}$$

Integrate: $$\log\frac{b}{b_0} = \int_{\lambda_0}^{\lambda}\frac{d\lambda}{\beta(\lambda)}$$

UV: small z, λ $$\beta(\lambda \to 0) = -c\lambda^2 \quad b(z \to 0) = \mathcal{L}/z$$

$$\lambda(z \to 0) = \frac{1}{c\log(1/\Lambda z)} \qquad \Lambda = \frac{b_0}{\mathcal{L}}$$

$$\phi(z \to 0) \sim \log\log z = -\infty$$

For all solutions, independent of f(z)!!

General strategy for getting p = p(T) and from this all the thermo

Find black hole solutions $f(z_h) = 0$ and vacuum solutions $f = 1$

Compute T, $S = A/(4G_5)$, $s = S/V$

Integrate $p = p(T)$ from $s(T) = p'(T)$

5 constants of integration:

$$
\begin{aligned}
&\lambda_h \equiv \lambda(z_h) \\
&b_h \quad \text{converted to } \Lambda, \quad \Lambda z = \text{ dimless} \\
&\dot{b}_h = b_h^3 V(\lambda_h)/(3\dot{f}_h) \text{ by 1st eq at } f = 0 \\
&f(0) = 1 \text{ for asymptotically AdS}_5 \\
&\dot{f}_h = -4\pi T
\end{aligned}
$$

For reference, a summary of numerical integration: 0903.2859,Appendix

Integration straightforward with NDSolve/Mathematica. Put the horizon at z_{start} + eps and compute analytically by expanding eqs the 5 initial conditions at z_{start} . Only λ_h remains unchanged under further scalings.

NDSolve produces a numerical soln in which b diverges, $\lambda=0$, f, W = some consts at some $z_1 < z_{start}$

Scale all the solns so that $W(z_1)=1$

Shift z so that z_1 goes to 0. W(0)=1 guarantees f(0)=1, b = 1/z

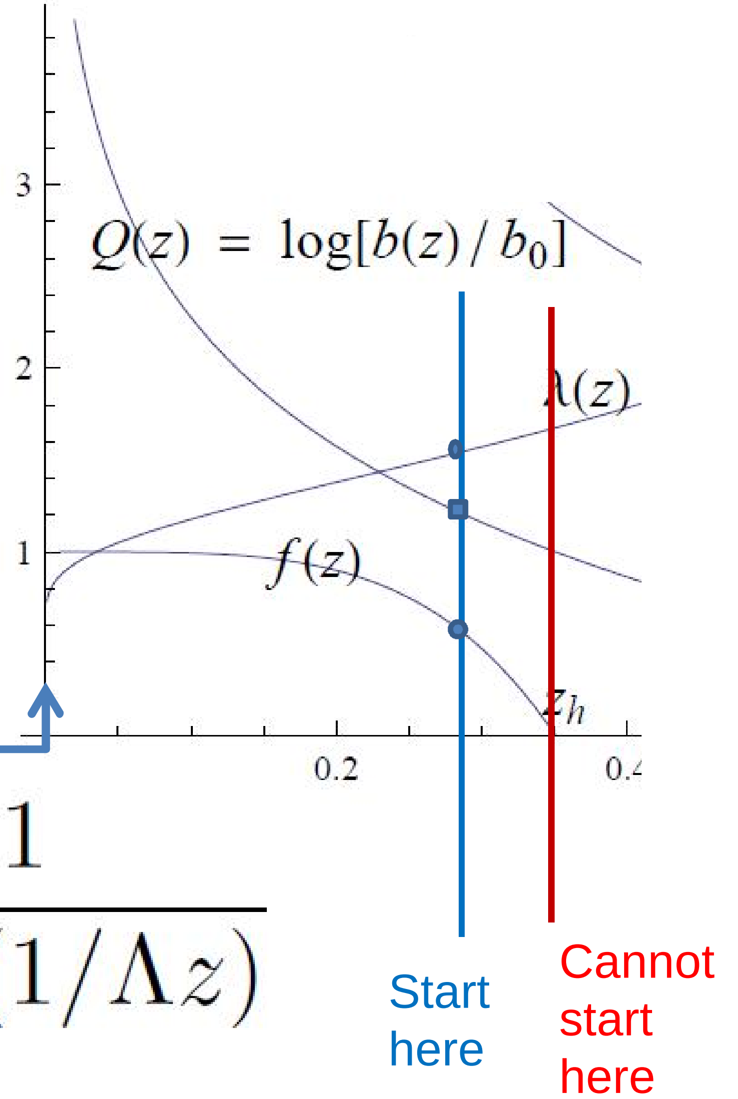
Enforce the const of integration Λ by scaling the z coordinate (and b,f,W in an appropriate way) so that at some very small λ, z with given Λ . Here $\beta(\lambda) = -\beta_0\lambda^2 + \ldots$

$$\lambda(z) = 1/(-\beta_0 \log(\Lambda z))$$

Then you have a BH soln with horizon somewhere and with some λ_h , Λ, T, S

$$W = -\dot{b}/b^2$$

$$\begin{aligned} \dot{W} &= 4bW^2 - \frac{1}{f}\left(W\dot{f} + \frac{1}{3}\,bV\right), \\ \dot{b} &= -b^2 W, \\ \dot{\lambda} &= \frac{3}{2}\,\lambda\sqrt{b\dot{W}}, \\ \ddot{f} &= 3\dot{f}bW, \end{aligned}$$



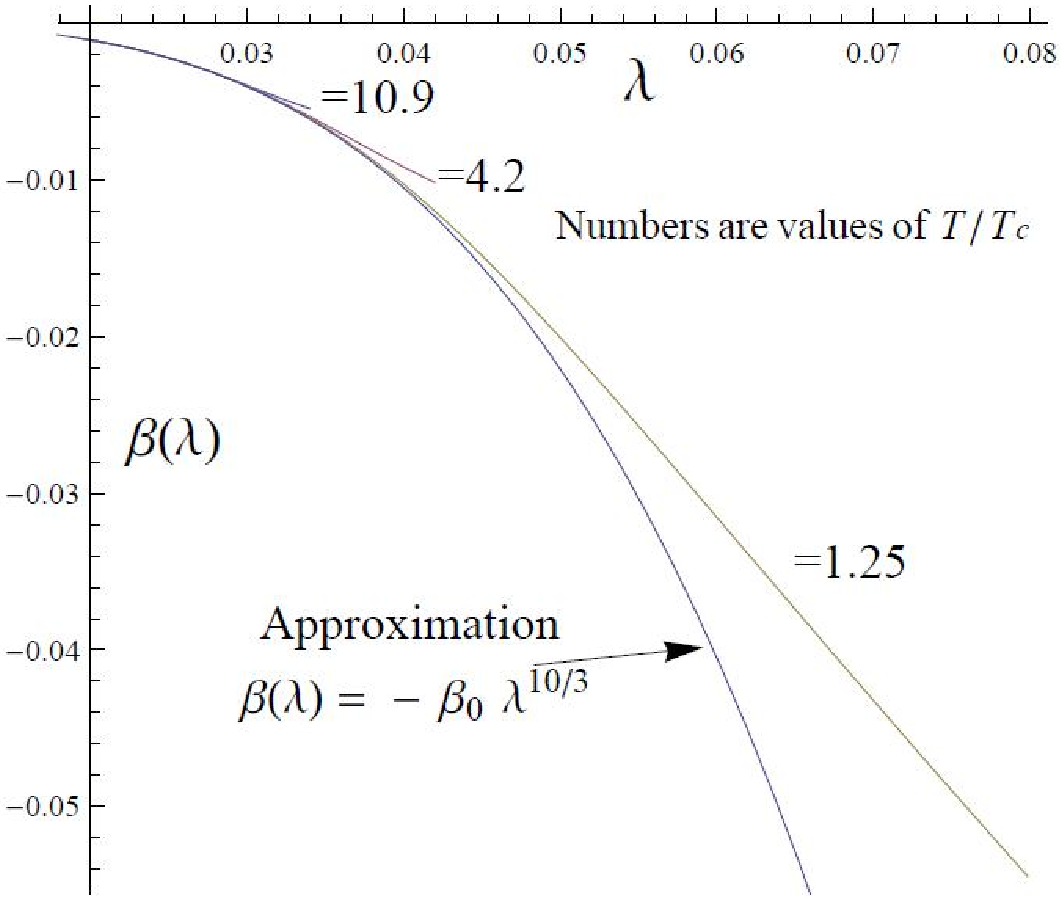
For all solns scale z so that

$$\frac{1}{c\log(1/\Lambda z)}$$

same Λ for all!

I

The potential $V(\phi) = \frac{12}{\mathcal{L}^2}\left\{1 + V_0\lambda + V_1\lambda^{4/3}[\log(1+V_3\lambda^2)]^{1/2}\right\}$ fits SU(N) thermo



Compute $b\frac{d\lambda}{db}$

from its solutions at various T

Can you get thermo from the approximation

$$\beta(\lambda) = -\beta_0\lambda^q$$

q = 10/3 ??

Beta functions: $\int \frac{d\lambda}{\beta(\lambda)}$ $\int \frac{\beta(\lambda)}{\lambda^2}$ should be analytically integrable

QCD-like
SU(N) Y-M

$$\beta(\lambda) = -\beta_0\lambda^q \qquad \beta(\lambda) = \frac{-\beta_0\lambda^q}{1+\frac{2}{3}\,\beta_0\lambda^{q-1}}$$

$$\beta(\lambda) \;=\; \frac{-\beta_0\lambda^q}{1+\frac{2}{3}\,\beta_0\lambda^{q-1}}\left[1+\alpha(q-1)\frac{\log(1+\frac{2}{3}\,\beta_0\lambda^{q-1})}{\log^2(1+\frac{2}{3}\,\beta_0\lambda^{q-1})+1}\right]$$

Logic of this monster: GKMN have shown that in the IR

$$\beta \to -\frac{3}{2}\,\lambda\left(1+\frac{\alpha}{\log\lambda}\right) \qquad \alpha > 0$$

in confining theories. We find q=10/3, α = ¼ gives good thermo

α = 0: continuous transition

Infrared
fixed point $\beta(\lambda) = -\beta_0\lambda^2\left(1-\frac{\lambda}{\lambda_*}\right)$

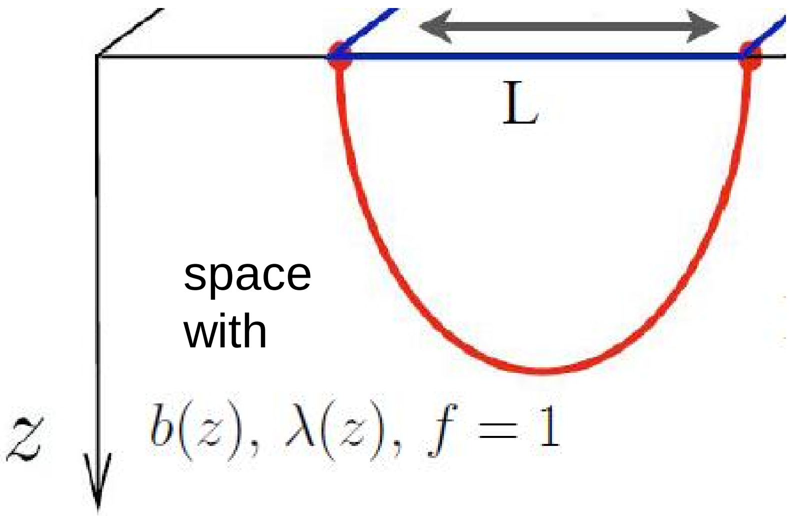
Walking
technicolor $\beta(\lambda) = -c\lambda^2\frac{(1-\lambda)^2+e}{1+a\lambda^3}$

Alanen-Kajantie 0912.4128

Alanen, talk
Alanen-Kajantie-Tuominen

How does confinement enter?

$$V(L) = \sigma L$$



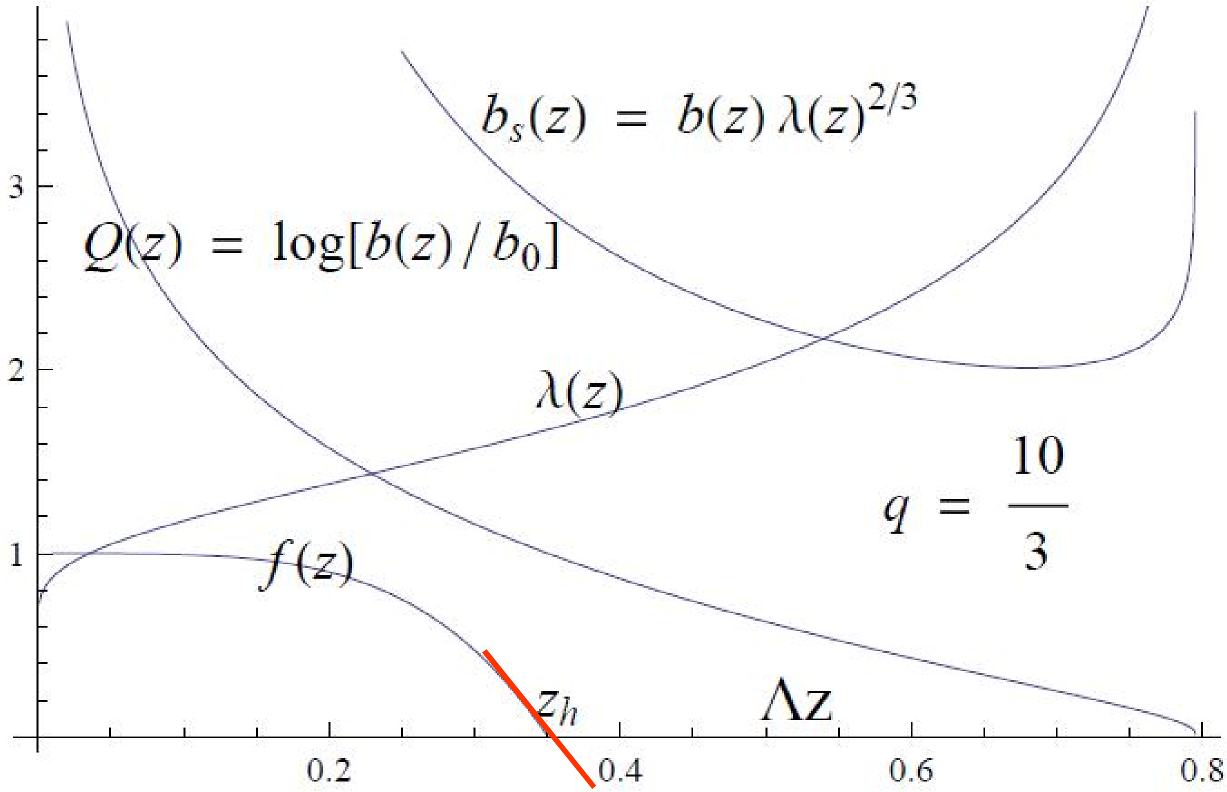
Condition for $L \to \infty$ is

$b(z)\lambda^{2/3}(z)$ have a minimum at some z_{min}

$$\frac{db}{b} + \frac{2}{3}\frac{d\lambda}{\lambda} = 0$$

$$\beta(\lambda_{\min}) = -\frac{3}{2}\lambda_{\min}$$!!

Typical field configs for $\beta \sim \lambda^{10/3}$ $\quad Q = \int^{\lambda} \frac{d\lambda}{\beta(\lambda)} = \frac{1}{(q-1)\beta_0 \lambda^{q-1}}$



$\pi T = \frac{1}{z_h} = \text{ big}$ $\quad 4\pi T = -f'(z_h)$ $\quad$ T big again!

T minimum!

UV $\quad$ IR

spherical BH $\pi T = \frac{1}{2r_h} + \frac{r_h}{\mathcal{L}^2}$

Some tricks for solving:

Introduce:

$$W = -\frac{\dot{b}}{b^2} \Rightarrow \frac{\ddot{b}}{b} - 2\frac{\dot{b}^2}{b^2} = -b\dot{W}$$

Since $\beta = b\dot{\lambda}/\dot{b}$ 2nd eq integrates to

$$W(\lambda) = W(0)\exp\left(-\frac{4}{9}\int_0^\lambda d\bar{\lambda}\frac{\beta(\bar{\lambda})}{\bar{\lambda}^2}\right) \qquad W(0) = 1/\mathcal{L}$$

Get b(z) by integrating def of W:

$$dz = \frac{db}{-b^2 W} \qquad \dim\frac{1}{z} \equiv \Lambda = \frac{b_0}{\mathcal{L}}$$

Put this to 1st eq:

$$V(\lambda) = 12fW^2\left[1 - \left(\frac{\beta}{3\lambda}\right)^2\right] - 3\frac{\dot{f}}{b}W$$

horizon position

With b(z) so computed:

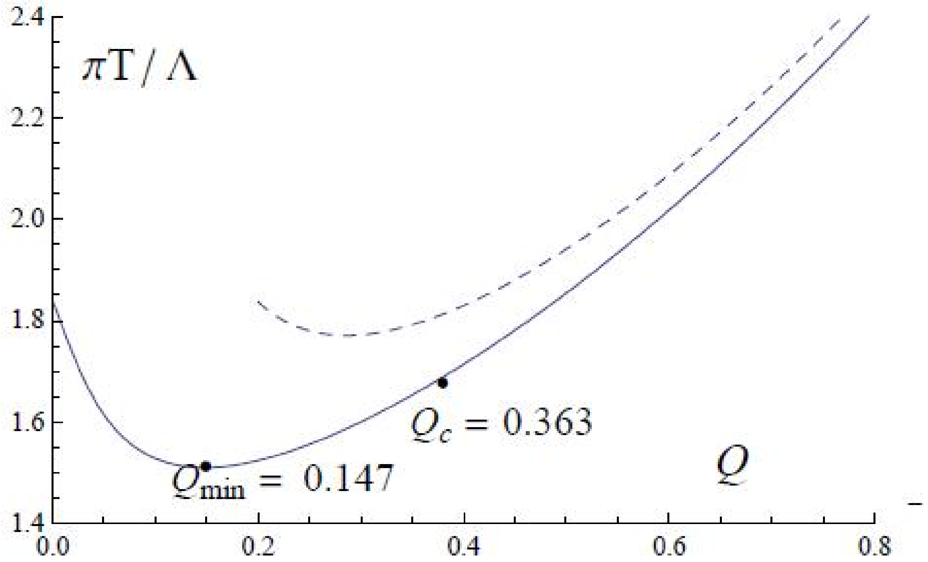
$$f(z) \;=\; 1-\int_0^z \frac{d\bar{z}}{b^3(\bar{z})}\Big/\int_0^{z_h}\frac{d\bar{z}}{b^3(\bar{z})}$$

$$\frac{1}{4\pi T} = b^3\int_0^z \frac{dz}{b^3} = b^3\int \frac{db}{-b^5 W} = \text{number } \Lambda$$

$$s = \frac{S}{V} = \frac{b^3}{4G_5}$$

For illustration, take the very simple $\beta(\lambda) = -\beta_0\lambda^q$ q = 10/3

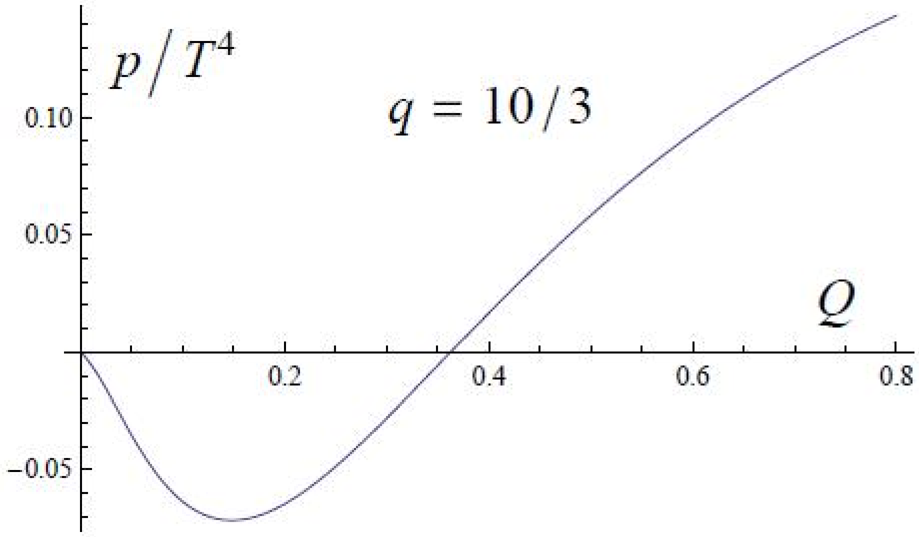
$$\log\frac{b}{b_0} = Q = \int^\lambda \frac{d\lambda}{\beta(\lambda)} = \frac{1}{(q-1)\beta_0\lambda^{q-1}}$$



$$p(T) = \int^T dT\, s(T)$$

$$\sim \int_0^Q dQ\, \frac{dT}{dQ}\, b^3(Q(T))$$

starts negative!!

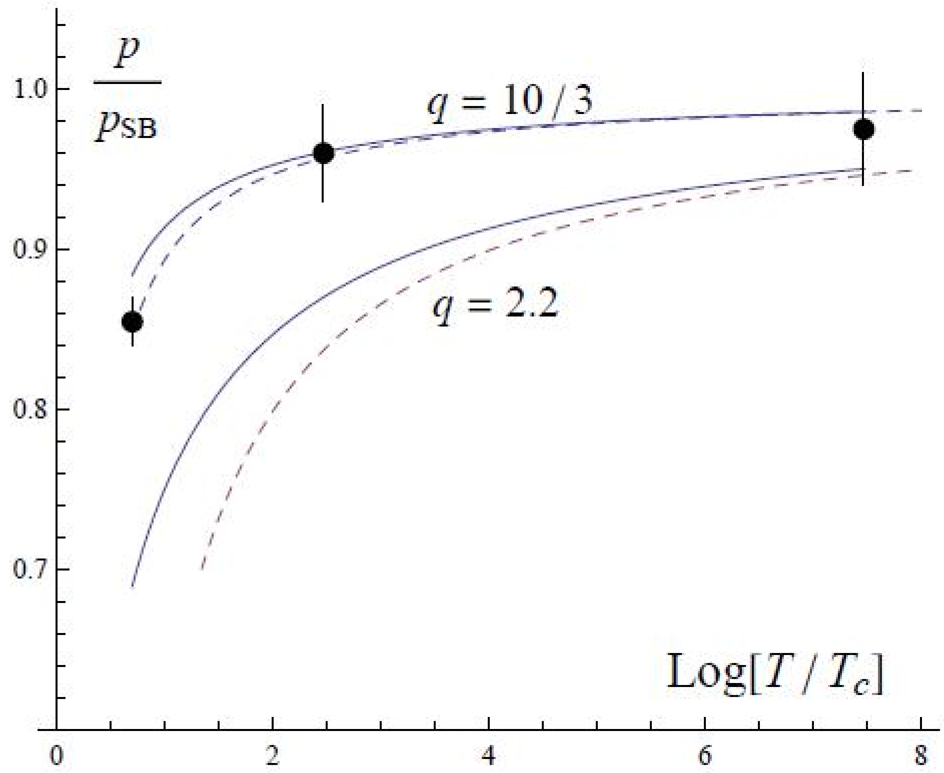


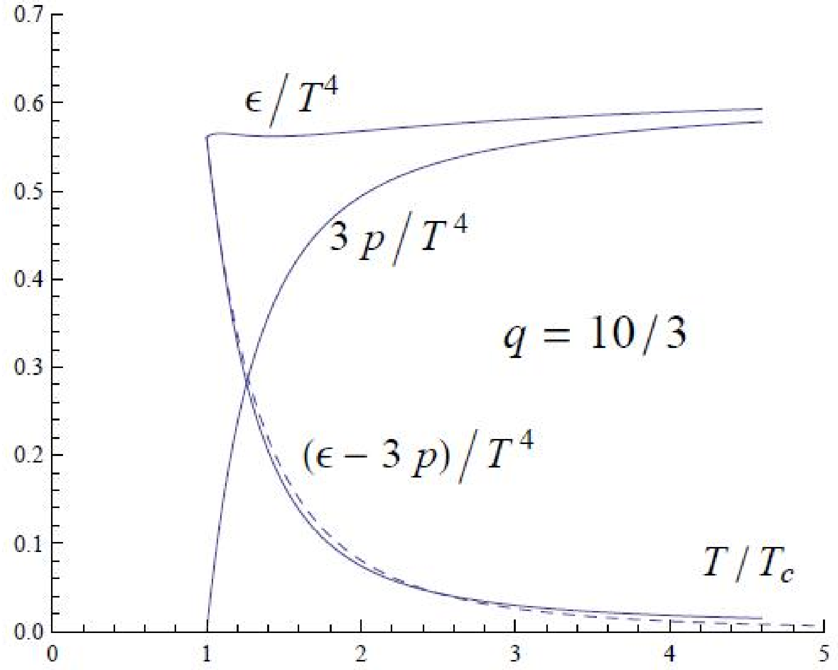
$$p(Q_c) = p(Q(T_c)) = 0$$

Now you have T_c !
but in units of Λ!

p/T^4 in units of $\mathcal{L}^3/G_5$

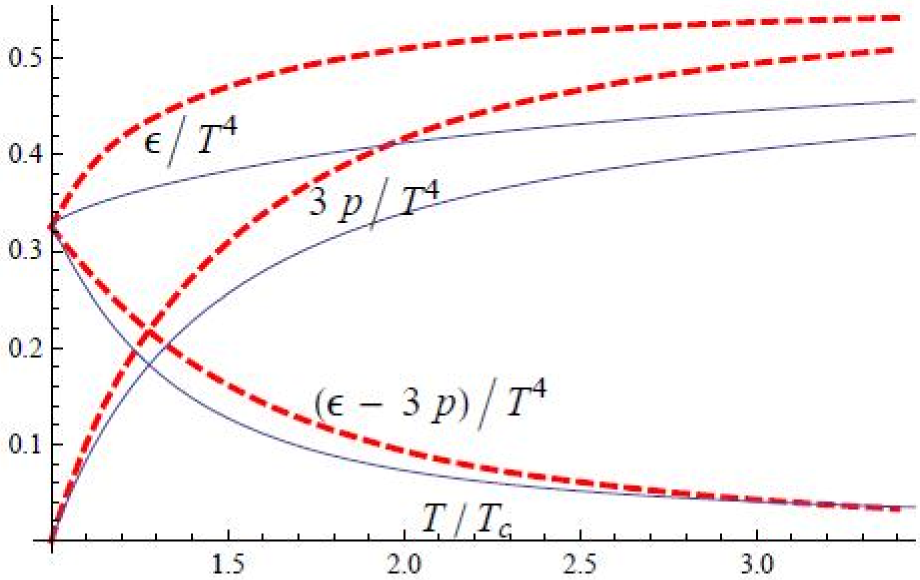
Large T:
q=10/3 looks fine but L too big!





$$\frac{\epsilon(T_c)}{N_c^2 T_c^4} \equiv \frac{L}{N_c^2 T_c^4}$$

too big, expect 0.34



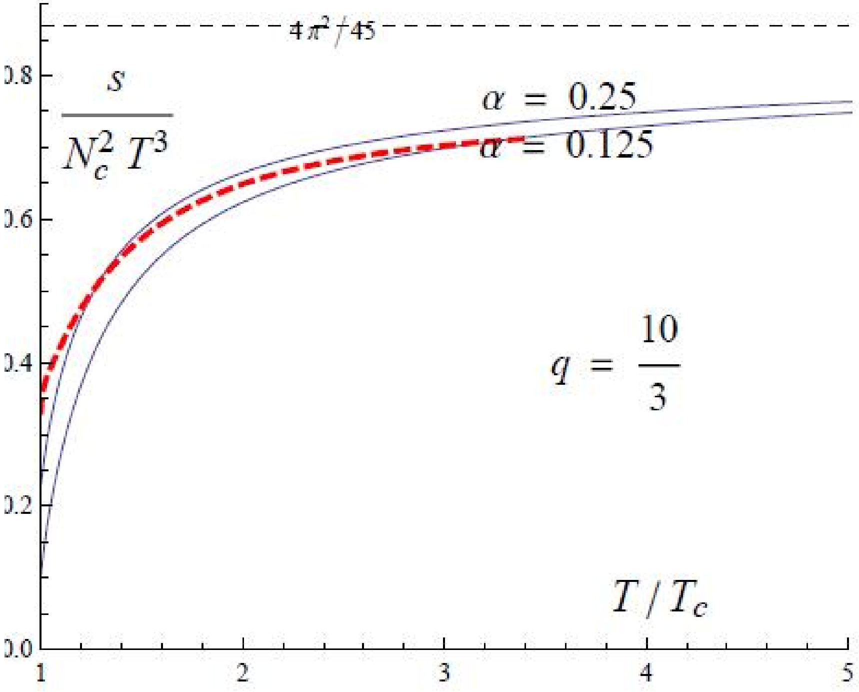
Fit q=2.2 to give correct L:

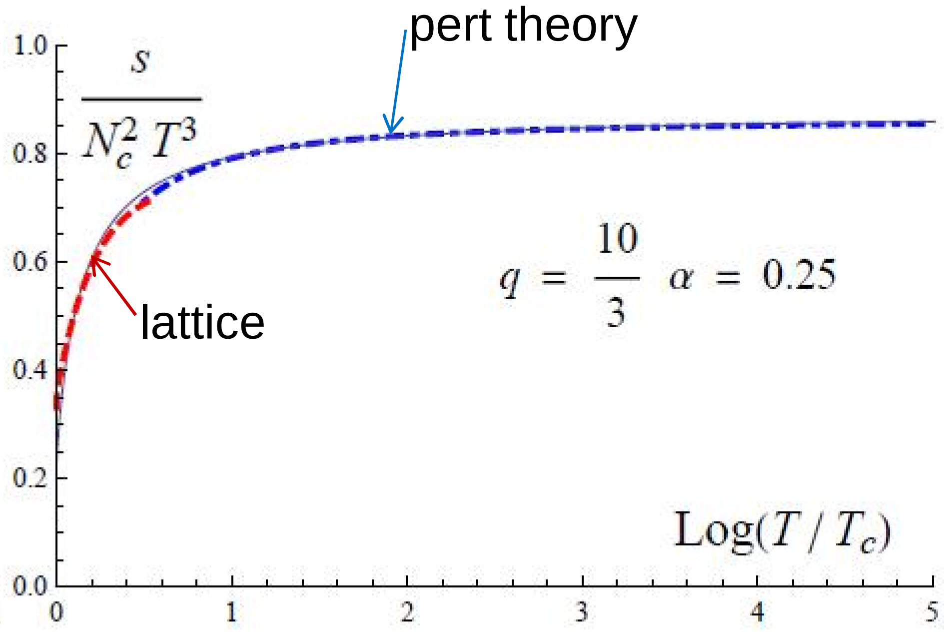
Red = SU(N_c) data/N^2_c

Panero 0907.3719

Almost right but one parameter not enough to get everything

For a good fit to SU(N) thermo need the monster beta fn
with 2 params (or the monster potential $\frac{12}{\mathcal{L}^2}\left\{1+V_0\lambda+V_1\lambda^{4/3}[\log(1+V_3\lambda^2)]^{1/2}\right\}$





7. Spatial string tension and gauge/gravity duality

Andreev-Zakharov, hep-ph/0607026, Andreev 0709.4395, AKS 0905.2032

V(L) computed from an x,y Wilson loop in the background b, λ, f:

$$= \frac{b_{s*}^2}{2\pi\alpha'}L + \frac{b_{s*}^2}{\pi\alpha'}\int_\epsilon^{z_*} dz \left[\frac{1}{\sqrt{f(z)}}\sqrt{\frac{b_s^4(z)}{b_{s*}^4} - 1} - \frac{b_s^2(z)}{b_{s*}^2}\right] + \frac{1}{\pi\alpha'}\int_\epsilon^{z_*} dz\, b_s^2(z)$$

Leading large-L piece = tension

Two-quark energy
= 2 Polyakov line

Noronha

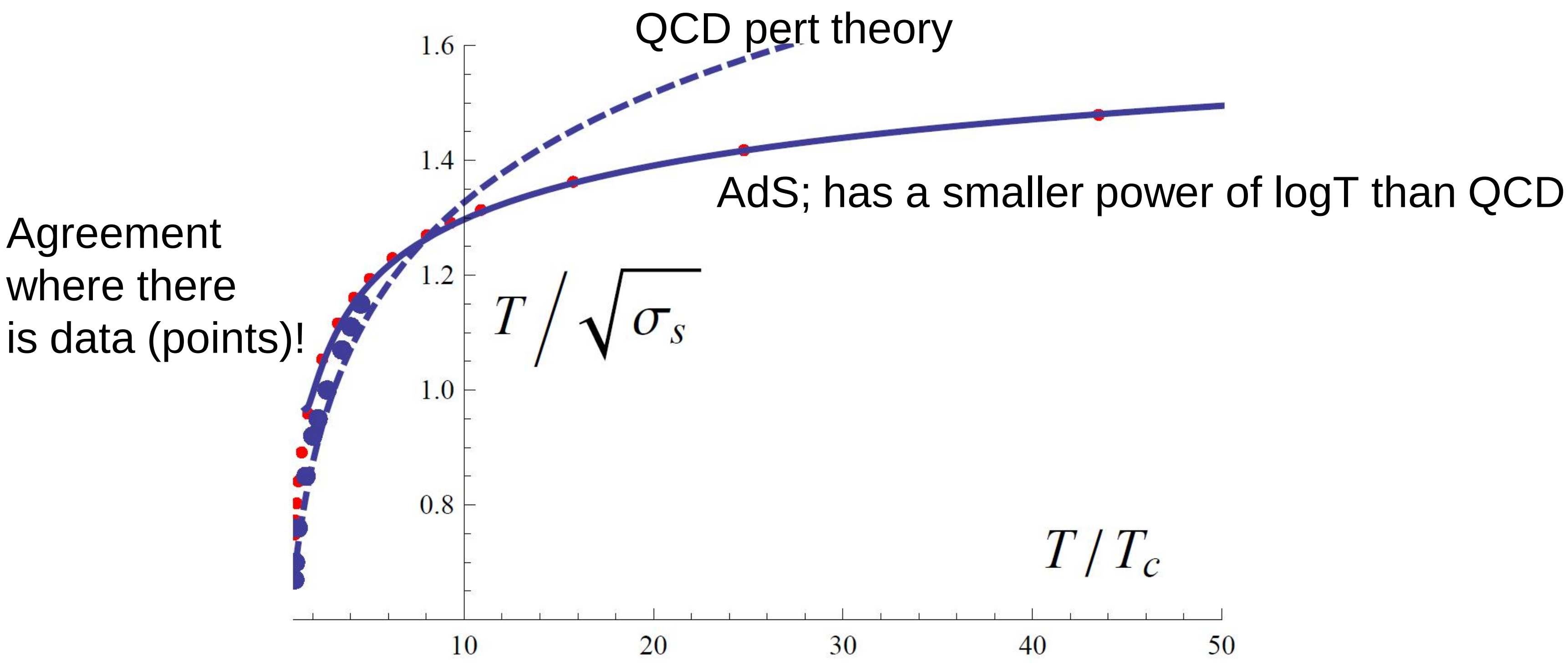
$$z_* \longrightarrow z_h \qquad \sigma_s = \frac{1}{2\pi\alpha'} b^2(z_h)\lambda^{4/3}(z_h)$$

Now you have T-dependence but magnitude = ? Can fit to data or take α' from fits to glueball masses Nitti Or relate to the usual string tension of YM theory at T=0:

$$\sigma = \frac{1}{2\pi\alpha'} b^2(z_{\min})\lambda^{4/3}(z_{\min})$$

Vacuum solns b(z), λ(z)

In all cases the outcome is:



8. Beyond QCD (Talk by Janne Alanen)

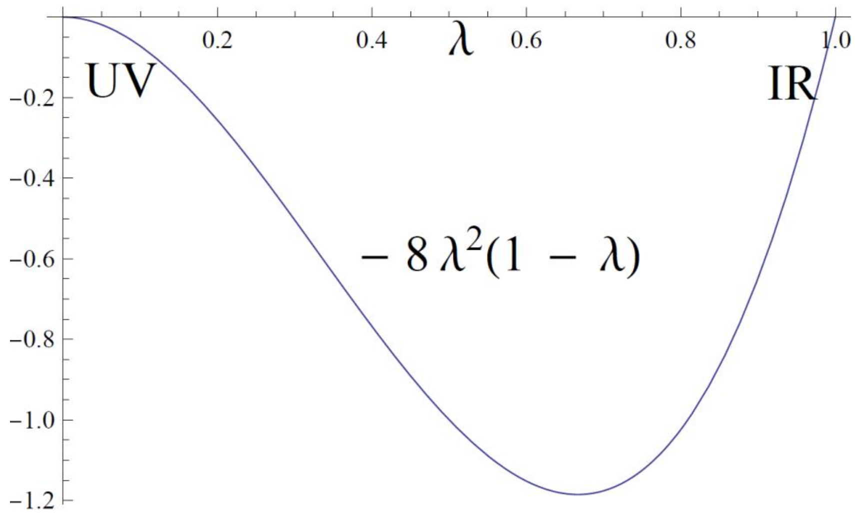
8.1 Beta function with infrared fixed point

Alanen-Kajantie 0912.4128

$$\beta(\lambda) = -c\lambda^2(1-\lambda)$$

Asymptotically AdS, $\mathcal{L}_{\mathrm{UV}}$
conformal, partonic phase

In the IR massless "unparticles"
conformal, another AdS $\mathcal{L}_{\mathrm{IR}}$

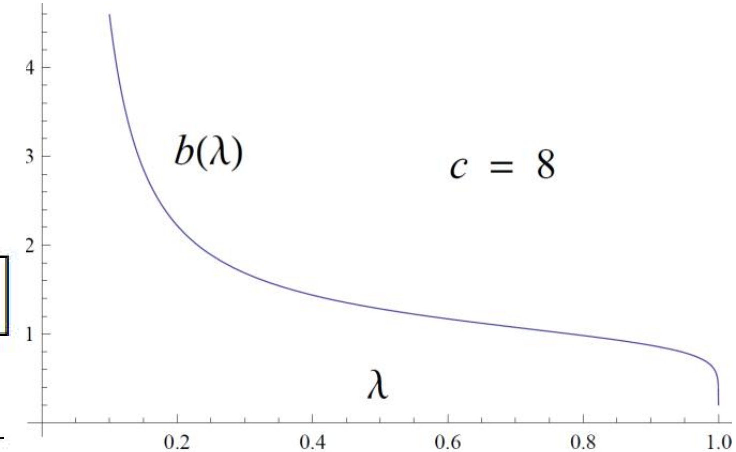


$$c = \frac{9}{2}\log\frac{\mathcal{L}_{\mathrm{UV}}}{\mathcal{L}_{\mathrm{IR}}}$$

Solve thermo using the algorithm given:

$$b(z) = b_0 \exp\left[\frac{1}{c\lambda} + \frac{1}{c}\log\left(\frac{1}{\lambda} - 1\right)\right]$$

$$W(\lambda) = \underbrace{\frac{1}{\mathcal{L}_{\rm UV}}\,\exp\left[\tfrac{2}{9}\,c\left(1-(1-\lambda)^2\right)\right]}_{\frac{1}{\mathcal{L}_{\rm IR}}} = -\frac{db}{b^2 dz}$$



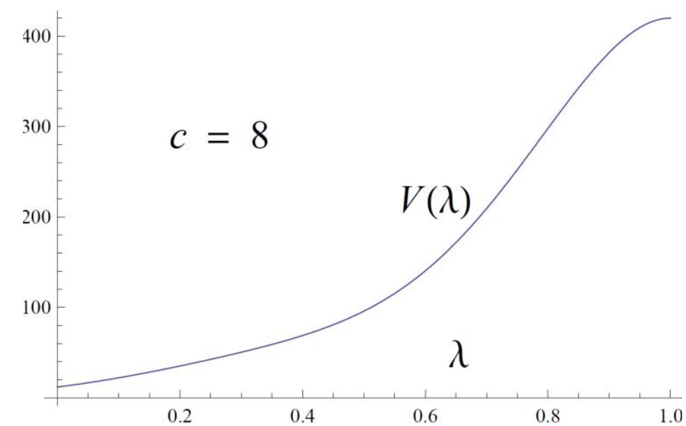
$$V(\lambda) = \frac{12}{\mathcal{L}_{\rm UV}^2}\,\exp\left[\tfrac{4}{9}\,c\left(1-(1-\lambda)^2\right)\right]\,\left[1 - \tfrac{1}{9}\,c^2\lambda^2(1-\lambda)^2\right]$$

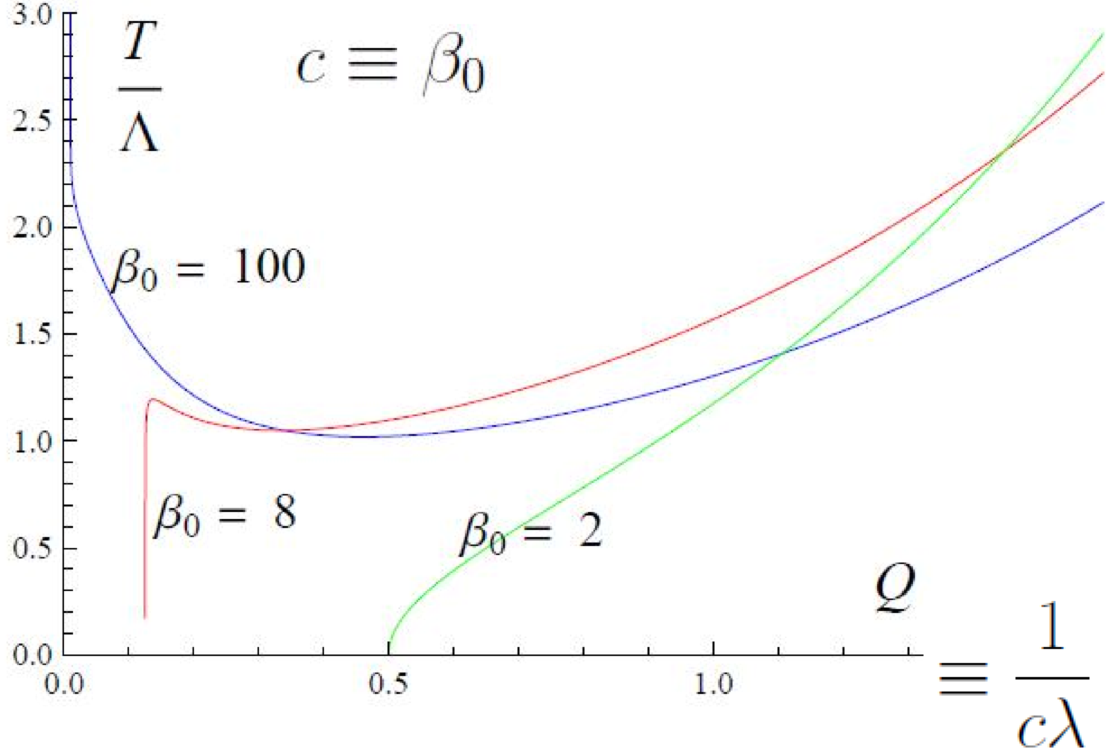
f (z)= 1

$$f(z) = 1 - \int_0^z \frac{d\bar{z}}{b^3(\bar{z})} \Big/ \int_0^{z_h} \frac{d\bar{z}}{b^3(\bar{z})}$$

$$\frac{1}{4\pi T} = b^3 \int_0^z \frac{dz}{b^3} = b^3 \int_0^\lambda \frac{d\lambda}{-\beta b^4 W}$$

$$s(T) = \frac{b^3(\lambda(T))}{4G_5}$$

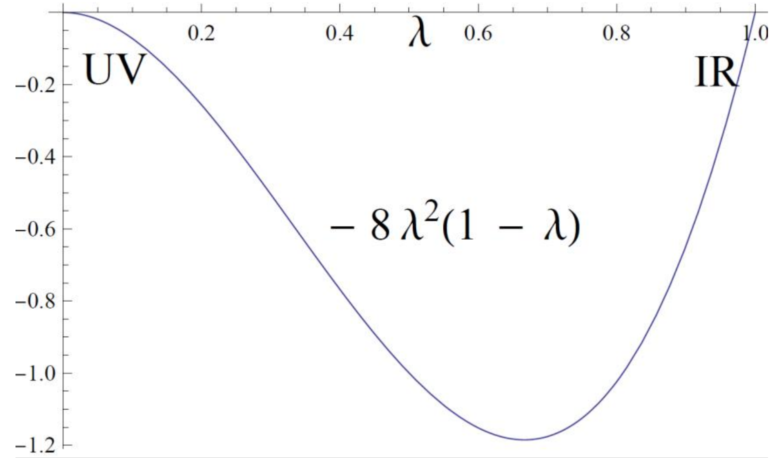


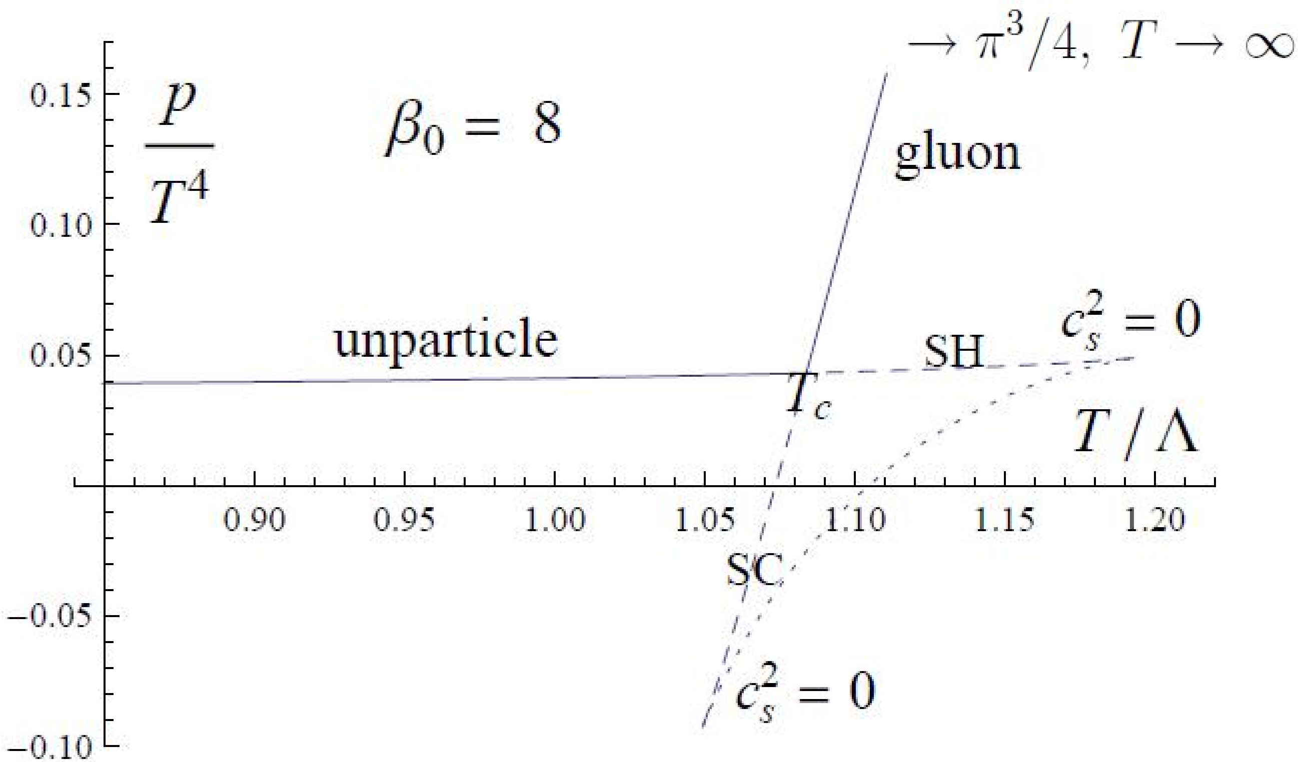


A minimum in T if c > 6.6: 1st order transition

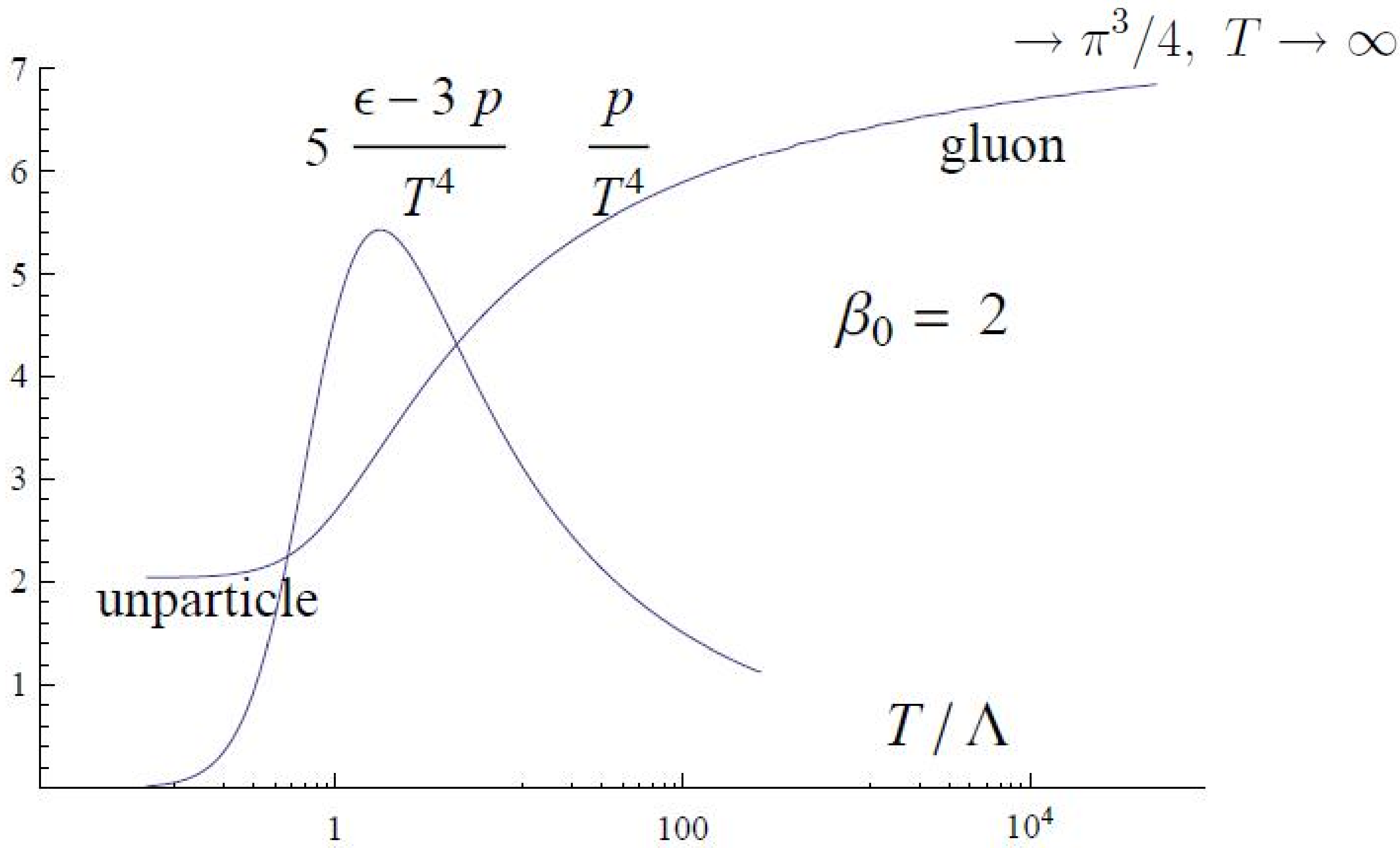
$\lambda = 0 \quad 0.26 \quad 0.37 \quad 0.91 \quad 0.99 \;\; 1.0$

"gluon"	super-cooled	unstable	super-heated	unparticle





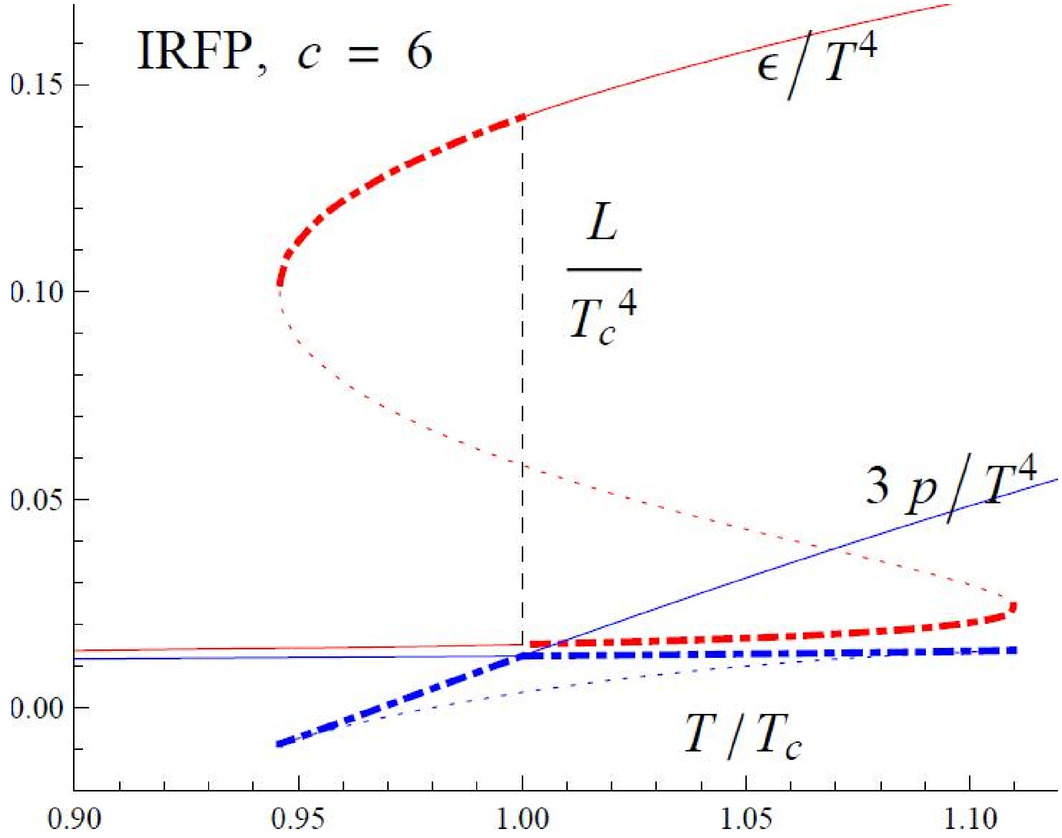
For c < 6.6 a continuous transition:



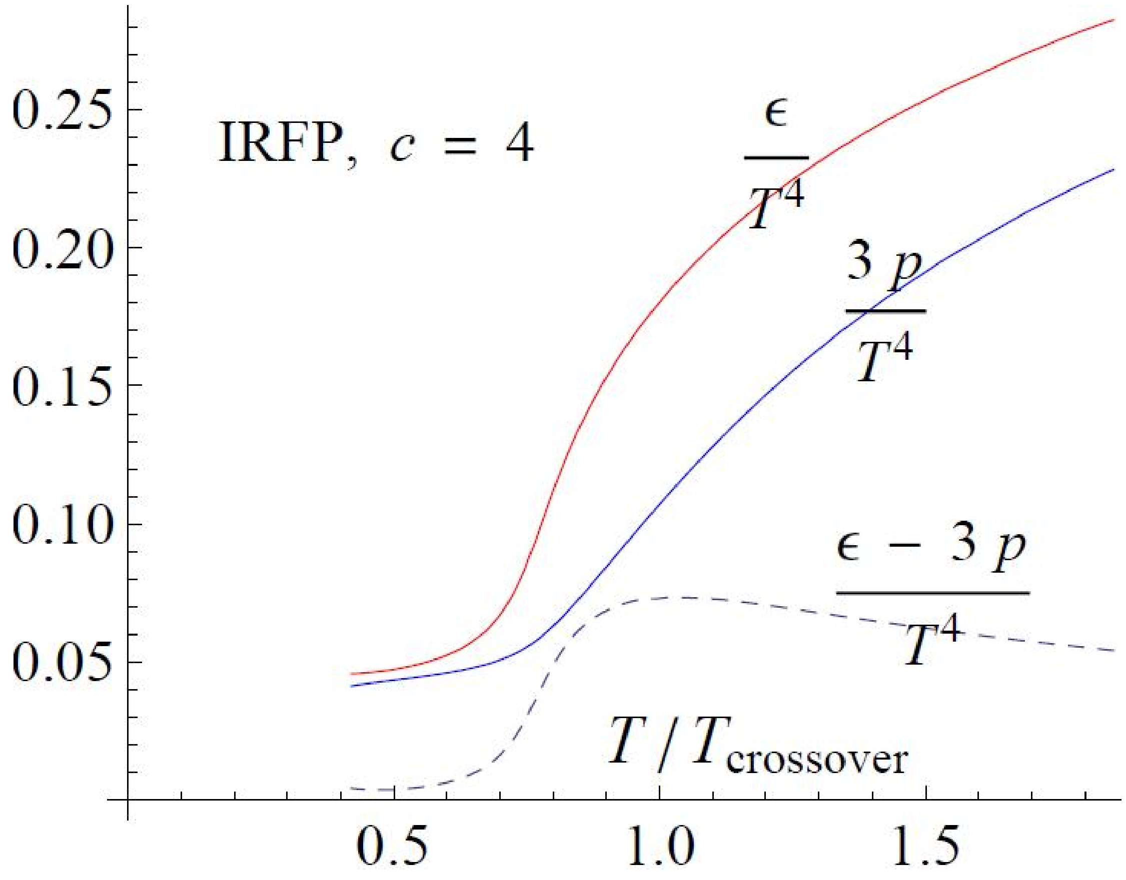
Obtain essentially the same by numerical integration with

$$V(\lambda) = \frac{12}{\mathcal{L}_{\rm UV}^2}\ \exp\left[\tfrac{4}{9}\,c\left(1-(1-\lambda)^2\right)\right]\ \left[1-\tfrac{1}{9}\,c^2\lambda^2(1-\lambda)^2\right]$$

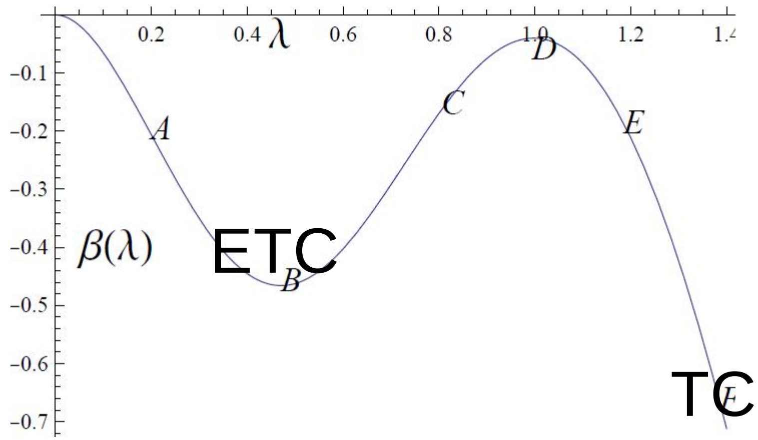
Produces a 1st order transition for c > 4.85:



a continuous transition for c < 4.85:



8.2 Walking technicolor:



~ EW

TC ~ 246 GeV
ETC ~ 1000 TC

To get big top quark mass while avoiding strangeness changing neutral currents the techniquark condensate must vary a lot:

$$\langle \bar{Q}Q(\mathrm{ETC})\rangle = \underbrace{\exp\left(\int_{\mathrm{TC}}^{\mathrm{ETC}} \frac{d\mu}{\mu}\,\gamma(g^2(\mu))\right)}\,\langle \bar{Q}Q(\mathrm{TC})\rangle$$

$$[\log(\mathrm{ETC}/\mathrm{TC})]^{\mathrm{const}}$$ coupling runs

$$(\mathrm{ETC}/\mathrm{TC})^{\gamma(g_*^2)}$$ coupling walks

Technicolor/Gravity now gives thermodynamics – in terms of unknown parameters !

Alanen-Kajantie-Tuominen

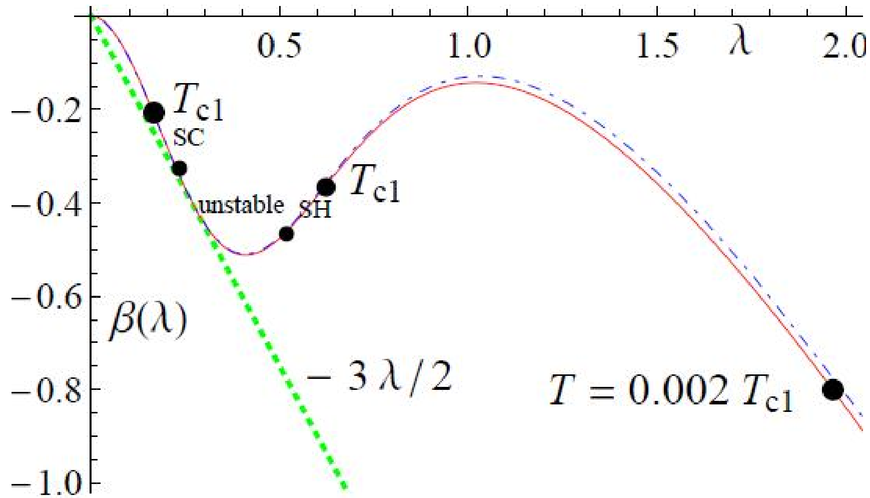
Take $$\beta(\lambda) = -c\lambda^2 \frac{(1-\lambda)^2 + e}{1 + a\lambda^3}$$

c = 9.68, a = 2/3 c so that $\beta(\lambda) > -\frac{3}{2}\lambda$ unless $\lambda >> 1$

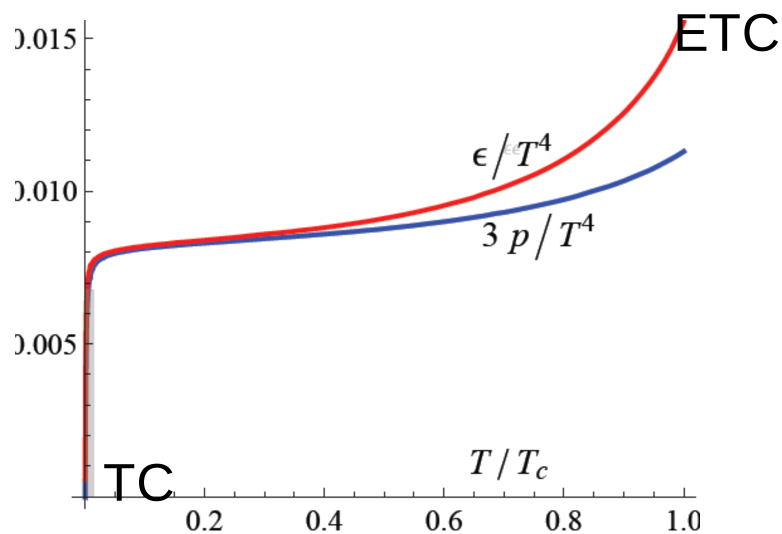
do not want confinement for small λ

$$V(\lambda) = 12\exp\left(-\tfrac{8}{9}\int_0^\lambda d\lambda \frac{\beta(\lambda)}{\lambda^2}\right)\left(1 - \frac{\beta^2}{9\lambda^2}\right) \times \left[1 + \tfrac{e}{10}\sqrt{\log(1+\lambda^4)}\right]$$

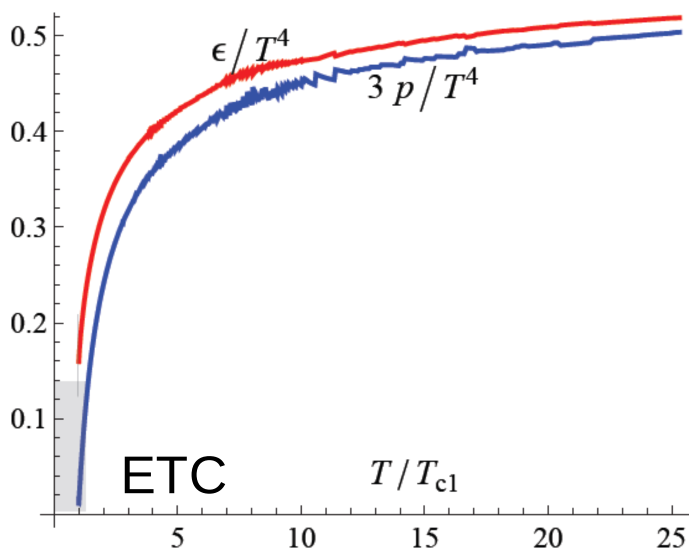
$$\sim \lambda^{4/3}\sqrt{\log\lambda}$$



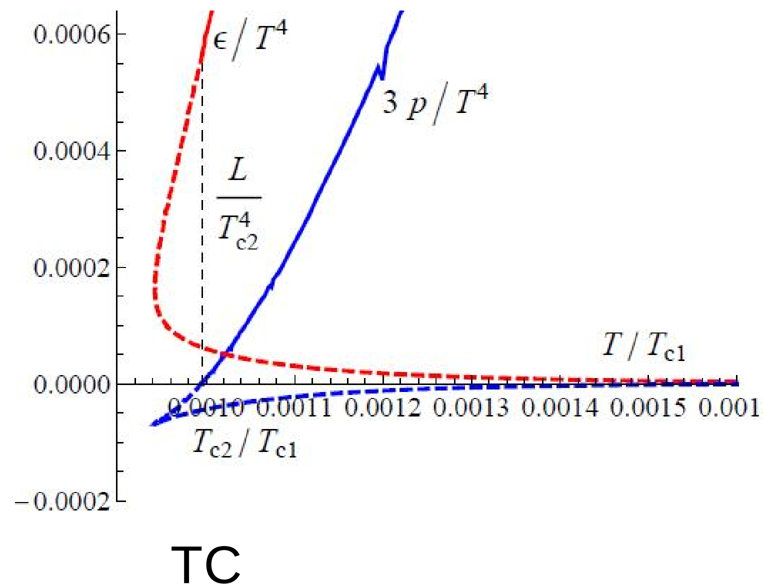
0.5
1.0
1.5
λ
2.0
−0.2
−0.4
−0.6
−0.8
−1.0
T_{c1}
SC
unstable
SH
T_{c1}
$\beta(\lambda)$
$-3\lambda/2$
$T = 0.002\,T_{c1}$



ETC
0.015
0.010
0.005
ϵ/T^4
$3p/T^4$
TC
T/T_c
0.2
0.4
0.6
0.8
1.0



0.5
0.4
0.3
0.2
0.1
ϵ/T^4
$3p/T^4$
ETC
T/T_{c1}
5
10
15
20
25



0.0006
0.0004
0.0002
0.0000
−0.0002
ϵ/T^4
$3p/T^4$
$\frac{L}{T_{c2}^4}$
T/T_{c1}
0.0010
0.0011
0.0012
0.0013
0.0014
0.0015
0.0016
T_{c2}/T_{c1}
TC