Heavy $\bar{q}q$ Free Energy and Thermodynamics in AdS/QCD

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H.J.Pirner & B.Galow PLB679(2009); H.J.Pirner & J.Nian NPA833(2010); B. Galow et al. 0911.0627(2090); K.Veschgini et al. 0911.1680(2009); E.Megias et al. in preparation (2010).

Issues

1. Motivation
2. Heavy $\bar{q}q$ potential at zero temperature
   - Scale Invariance and Confinement
   - Soft-wall model of AdS/QCD
   - The 5D Einstein-dilaton model
3. Thermodynamics of AdS/QCD
   - Black Holes
   - The 5D Einstein-dilaton model at finite temperature
   - Thermodynamics
4. Heavy $\bar{q}q$ Free Energy
   - Polyakov Loop
   - Heavy $\bar{q}q$ free energy at $T > T_c$
   - Spatial Wilson Loops
   - Heavy $\bar{q}q$ free energy at $T \leq T_c$
5. Conclusions

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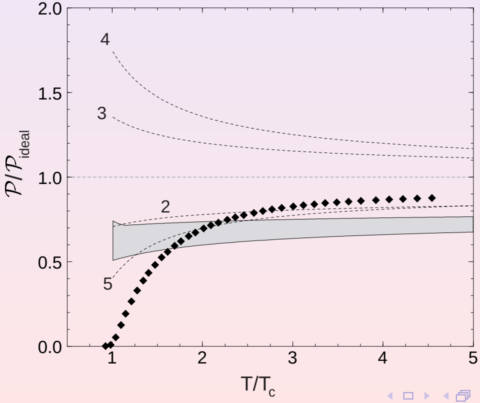
Motivation

Pressure of Gluodynamics

Weak Coupling Expansion and Resummed Perturbation Theory

E. Braaten and A. Nieto (1996), J.O. Andersen et al (1999).



Motivation

Interaction Measure in Gluodynamics

Weak Coupling Expansion and **Resummed Perturbation Theory**

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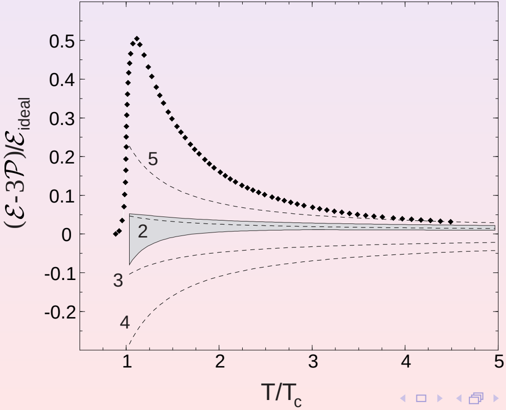
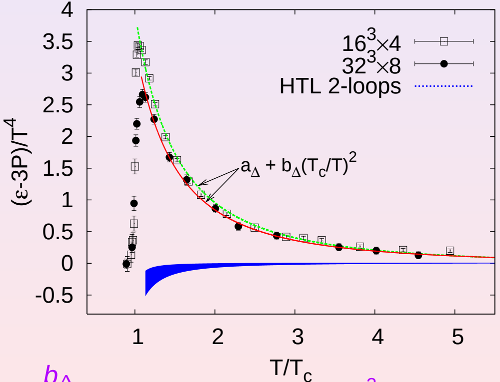


Motivation

Trace Anomaly $N_c = 3, N_f = 0$
G. Boyd et al., Nucl. Phys. B469, 419 (1996).



$$\frac{\epsilon - 3P}{T^4} = a_\Delta + \frac{b_\Delta}{T^2}\,, \quad b_\Delta = (3.46 \pm 0.13)T_c^2\,, \qquad 1.13T_c < T < 4.5T_c\,.$$

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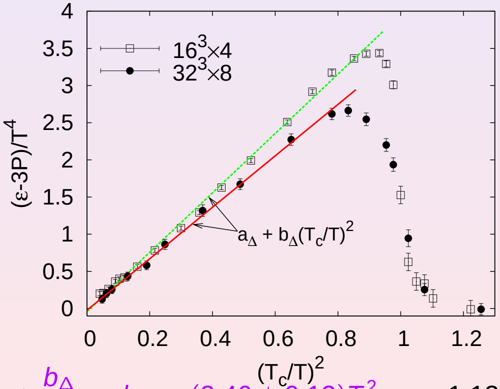
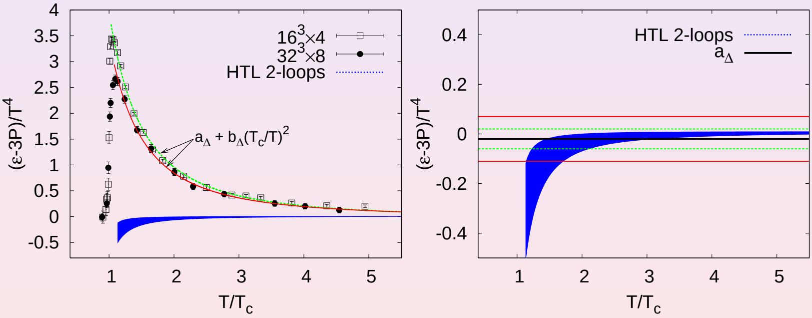


$$\frac{\epsilon - 3P}{T^4} = a_\Delta + \frac{b_\Delta}{T^2}\,, \quad b_\Delta = (3.46 \pm 0.13) T_c^2\,, \qquad 1.13 T_c < T < 4.5 T_c\,.$$

Motivation

$$\Delta \equiv \frac{\epsilon - 3P}{T^4} = a_{\Delta,P} + \frac{b_\Delta}{T^2}$$



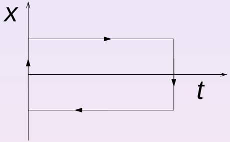
Perturbation Theory and Hard Thermal Loops only yield a_Δ !!.

Issues

Scale Invariance and Confinement

Consider a rectangular Wilson loop:

$$W(\mathcal{C}) = \exp\left(ig \int_{\mathcal{C}} A_\mu dx^\mu \right)$$



It is related to the potential $V_{q\bar{q}}(R)$ acting between charges q and $\bar{q}$:

$$W(\mathcal{C}) \to \exp\left(-T \cdot V_{q\bar{q}}(R)\right)$$

Scale transformations: $T \to \lambda T$, $R \to \lambda R$,
The only scale invariant solution is the Coulomb Potential:

$$V_{q\bar{q}} \sim \frac{1}{R}$$

Running coupling and string tension break scale invariance:

$$V_{q\bar{q}}(r) = -\frac{4}{3}\frac{\alpha_s(R)}{R} + \sigma R \,.$$

Issues

Soft-wall model of AdS/QCD

$$ds^2_{\rm QCD} = h(z) \cdot ds^2 = h(z) \frac{L^2}{z^2} \langle -dt^2 + d\vec{x}^2 + dz^2 \rangle \,.$$

- $h(z) = 1 \Longrightarrow$ Conformal
- $h(z) \neq 1 \Longrightarrow$ Non conformal

Breaking of scaling invariance in QCD is given by the running coupling:

$$\Delta \equiv \frac{\epsilon - 3p}{T^4} = \frac{\beta(\alpha_s)}{4\alpha_s^2} \langle F^2_{\mu\nu} \rangle \,.$$

where $\beta(\alpha_s) = \mu \frac{d\alpha_s}{d\mu}$ and $\alpha_s(E) \sim 1/\log(E/\Lambda)$.

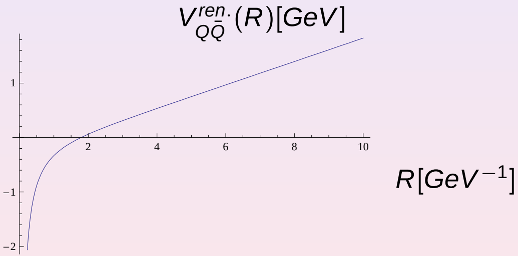
$\Longrightarrow$Assume an ansatz for conformal invariance breaking similar to 1-loop running coupling (H.J.Pirner & B. Galow '09):

$$h(z) = \frac{\log(\epsilon)}{\log(\epsilon + (\Lambda z)^2)} \,, \qquad z \sim \frac{1}{E} \,.$$

Soft-wall model of AdS/QCD

H.J.Pirner & B.Galow '09

$$W(\mathcal{C}) \approx \exp\left(-S_{\mathrm{NG}}\right) \propto \exp\left(-T \cdot V_{q\bar{q}}(R)\right)$$



$$V_{\bar{q}q} \approx -\frac{a}{R} + \sigma R\,, \qquad a = 0.48\,, \quad \sigma = (0.425\,\mathrm{GeV})^2$$

$$\Longrightarrow \qquad \epsilon = \Lambda^2 l_s^2 = 0.48\,, \qquad \Lambda = 264\,\mathrm{MeV}\,.$$

Issues

The 5D Einstein-dilaton model

5D Einstein-dilaton model (Gürsoy et al. '08):

$$S_{5D} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R - \frac{4}{3} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) - \frac{1}{8\pi G_5} \int_{\partial M} d^4x \sqrt{-h}\, K \,.$$

One to one relation between β-function and dilaton potential $V(\phi)$:

$$V(\phi) = V_0 \left(1 - \left(\frac{\beta(\alpha)}{3\alpha} \right)^2 \right) \exp\left[-\frac{8}{9} \int_0^\alpha \frac{\beta(a)}{a^2} da \right] , \qquad \alpha = e^\phi \,.$$

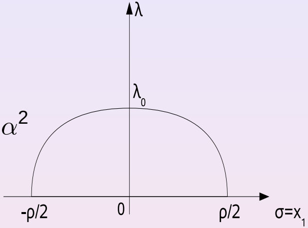
Ansatz:

$$\beta(\alpha) = -b_2\alpha + \left[b_2\alpha + \left(\frac{b_2}{\bar{\alpha}} - b_0 \right) \alpha^2 + \left(\frac{b_2}{2\bar{\alpha}^2} - \frac{b_0}{\bar{\alpha}} - b_1 \right) \alpha^3 \right] e^{-\alpha/\bar{\alpha}} \,.$$

- $\alpha << \bar{\alpha} \Longrightarrow$ Ultraviolet: $\beta(\alpha) \approx -b_0\alpha^2 - b_1\alpha^3$
- $\alpha >> \bar{\alpha} \Longrightarrow$ Infrared: $\beta(\alpha) \approx -b_2\alpha$

The 5D Einstein-dilaton model

$$ds^2 = e^{\frac{4}{3}\phi} e^{2A}\left(dt^2 + d\vec{x}^2\right) + e^{\frac{4}{3}\phi}\frac{12}{V_0} e^{2D} d\alpha^2$$



$$\rho(\alpha_0) = \int_{-\frac{\rho}{2}}^{\frac{\rho}{2}} d\sigma = \frac{12}{\sqrt{3V_0}} e^{-A_0} \cdot \int_0^{\alpha_0} \frac{e^{D-3\tilde{A}} \cdot \tilde{\alpha}^{-\frac{4}{3}}}{\sqrt{1-\tilde{\alpha}^{-\frac{8}{3}} e^{-4\tilde{A}}}} d\alpha\,,$$

$$\begin{aligned} V_{Q\bar{Q}}^{\text{reg.}}(\alpha_0) &= \frac{1}{T} S_{\text{NG}}^{\text{reg.}} = V - V_s \\ &= \frac{12\alpha_0^{\frac{4}{3}} e^{A_0}}{\pi l_s^2 \sqrt{3V_0}} \left[\int_0^{\alpha_0} d\alpha \frac{\tilde{\alpha}^{\frac{4}{3}} e^{D+\tilde{A}}}{\sqrt{1-\tilde{\alpha}^{-\frac{8}{3}} e^{-4\tilde{A}}}} - \int_0^{\infty} d\alpha\, \tilde{\alpha}^{\frac{4}{3}} \cdot e^{D+\tilde{A}} \right] \end{aligned}$$

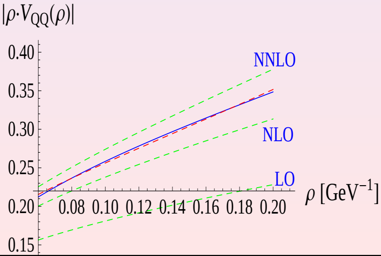
Heavy $\bar{Q}Q$ potential at Short Distances

Analytical result at short distances:

$$V_{Q\bar{Q}}(\rho) = -\frac{24}{\pi V_0 \bar{l}_s^2}\frac{\alpha_0^{4/3}(\rho)}{\rho}\Big[\underbrace{0.359}_{\text{LO}} + \underbrace{0.533 b_0 \alpha_0}_{\text{NLO}} + \underbrace{(1.347 b_0^2 + 0.692 b_1)\alpha_0^2}_{\text{NNLO}}\Big]$$

where the running coupling is

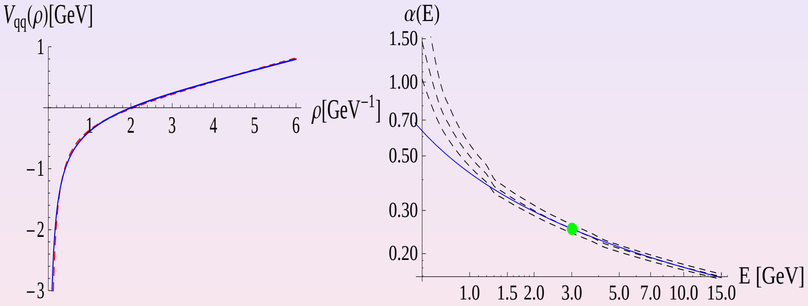
$$\alpha_0(\rho) = \left[b_0 \log\left(\frac{4.57}{\rho\sqrt{V_0}}\right) + \frac{b_1}{b_0}\log\left(b_0 \log\left(\frac{4.57}{\rho\sqrt{V_0}}\right)\right)\right]^{-1} + \mathcal{O}\left(\log^{-3}\right)$$



Comparison with PT $\Longrightarrow$

$$V_0 \bar{l}_s^2 = 1.31$$

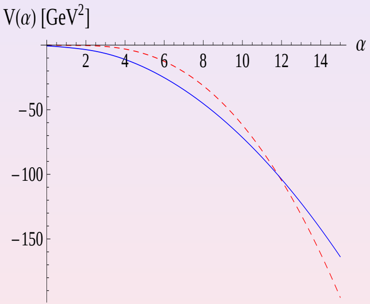
Heavy $\bar{Q}Q$ potential and running coupling

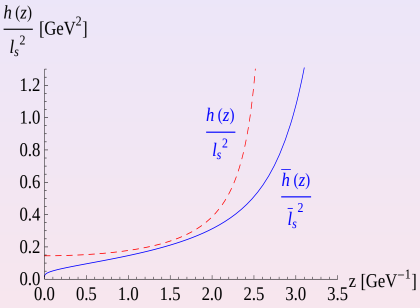


- $\sigma = (0.425\,\mathrm{GeV})^2 \Longrightarrow \frac{b_2}{\bar{\alpha}} = 3.51\mathrm{GeV} \cdot \bar{l}_s$
- Fit of running coupling $\Longrightarrow \frac{b_2}{\bar{\alpha}} = 5.09$

$$\bar{l}_s = 1.45\,\mathrm{GeV}^{-1}$$

Dilaton potential and warp factor

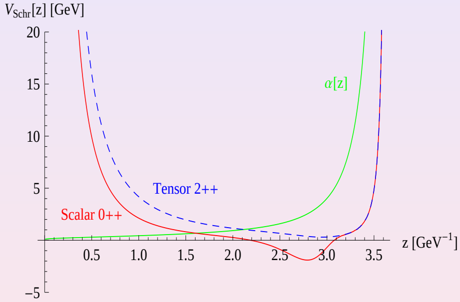




$$V(\alpha) \sim (b_2^2 - 9)\alpha^{\frac{8}{9}b_2}\,, \quad \alpha \to \infty; \qquad \alpha(z) \sim \frac{1}{(\bar{z}_{\mathrm{IR}} - z)^{\frac{b_2}{\frac{4}{9}b_2^2 - 1}}}\,, \quad z \to \bar{z}_{\mathrm{IR}}$$

Confinement and 'good' IR singularity

Effective Schrödinger potentials for glueballs 0^{++} and 2^{++}:



- Confining theory $\Longrightarrow 1.5 < b_2$
- IR singularity repulsive to physical modes $\Longrightarrow b_2 < 2.37$

Best choice of parameters:

$b_2 = 2.3\,, \quad \bar{\alpha} = 0.45\,, \quad V_0 = -0.623\,\mathrm{GeV}^2\,, \quad \bar{l}_s = 1.45\,\mathrm{GeV}^{-1}\,.$

Issues

Schwarzschild black hole

General Relativity with no source $\Longrightarrow$ Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g}\, R \,, \qquad R = g^{\mu\nu} R_{\mu\nu}$$

Classical solution $\frac{\delta}{\delta g_{\mu\nu}} \Longrightarrow$ Einstein equations

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \underset{\text{spherical}}{\Longrightarrow} R_{\mu\nu} = 0$$

Schwarzschild solution in spherical coordinates (1915):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_2^2 \,, \qquad f(r) = 1 - \frac{r_h}{r}$$

r_h is the horizon. Not physical singularity: $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 12 \frac{r_h^2}{r^6}$.
Large distance limit: $g_{tt}(r) \underset{r\to\infty}{\sim} -(1 + 2V_{Newton}(r)) \Longrightarrow r_h = 2G_4 M$.

Black hole thermodynamics

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) , \qquad \beta = \frac{1}{T}$$

Periodicity in euclidean time ($\tau = it$): $\Phi(\tau + \beta) = \Phi(\tau)$

- Regularity: Expansion around the horizon $r = r_h(1 + \rho^2)$:

$$ds^2 = 4r_h^2\left(d\rho^2 + \rho^2 \underbrace{\left(\frac{d\tau}{2r_h}\right)^2}_{d\theta^2} + \frac{1}{4}d\Omega_2^2\right)$$

$\Longrightarrow$ Periodicity: $\frac{\tau}{2r_h} \to \frac{\tau}{2r_h} + 2\pi \Longrightarrow \tau \to \tau + 4\pi r_h =: \tau + \beta$

$$T = \frac{1}{8\pi M G_4}$$

Thermodynamics interpretation of black holes:

$$dM = TdS \Longrightarrow S = \int \frac{dM}{T} = 4\pi G_4 M^2$$

Black hole thermodynamics

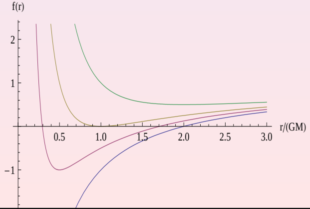
Area of the event horizon: $\mathcal{A} = 4\pi r_h^2 = 16\pi (G_4 M)^2$.
Bekenstein-Hawking entropy formula:

$$S = \frac{\mathcal{A}}{4G_D}$$

- Reissner-Nordström black hole:

$$S_{\mathrm{Einstein-Maxwell}} = \int d^D x \sqrt{-g} \left(\frac{1}{16\pi G_D} R - \frac{1}{4} F_{\mu\nu}^2 \right)$$

Solution: $f(r) = 1 - \frac{2G_4 M}{r} + \frac{G_4 Q^2}{r^2}$.



$Q^2 > G_4 M^2$ no singularity
$Q^2 = G_4 M^2$ extremal, T=0
$Q^2 < G_4 M^2$ two singularities
$Q^2 = 0$ one singularity

Issues

The 5D Einstein-dilaton model

$$S_{5D} = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R - \frac{4}{3}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) - \frac{1}{8\pi G_5}\int_{\partial M} d^4x\sqrt{-h}\,K$$

Finite temperature solutions (E. Kiritsis et al. JHEP (2009) 033):

- Thermal gas solution (confined phase):

$$ds^2_{\rm th} = b_0^2(z)(-dt^2 + d\vec{x}^2 + dz^2)\,, \qquad t \sim t + i\beta$$

- Black hole solution (deconfined phase):

$$ds^2_{\rm BH} = b^2(z)\left[-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right]$$

In the UV ($z \simeq 0$): flat metric $b(z) \simeq L/z$ and $f(0) = 1$.
There exists an horizon $f(z_h) = 0$.
Regularity at the horizon $\Longrightarrow T = \frac{|\dot{f}(z_h)|}{4\pi}$.

The 5D Einstein-dilaton model

Einstein equations $\frac{\delta}{\delta g_{\mu\nu}}$:

$$\underbrace{\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)}_{E_{\mu\nu}} - \underbrace{\left(\frac{4}{3}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu}\left(\frac{4}{3}(\partial\phi)^2 + V(\phi)\right)\right)}_{T_{\mu\nu}} = 0$$

$$(a) \qquad \frac{\ddot{f}}{\dot{f}} + 3\frac{\dot{b}}{b} = 0\,, \Longrightarrow f(z) = 1 - \frac{\int_0^z \frac{d\bar{z}}{b(\bar{z})^3}}{\int_0^{z_h} \frac{d\bar{z}}{b(\bar{z})^3}}$$

$$(b) \qquad 6\frac{\dot{b}^2}{b^2} - 3\frac{\ddot{b}}{b} = \frac{4}{3}\dot{\phi}^2\,,$$

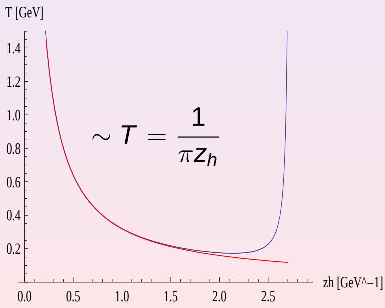
$$(c) \qquad 6\frac{\dot{b}^2}{b^2} + 3\frac{\ddot{b}}{b} + 3\frac{\dot{b}}{b}\frac{\dot{f}}{f} = \frac{b^2}{f} V(\phi)$$

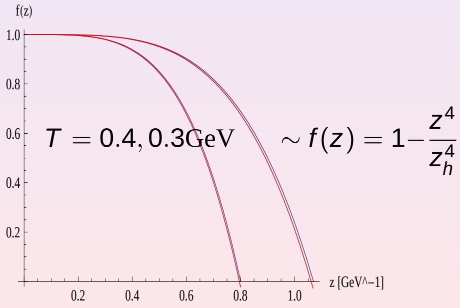
Conformal solution:

$$V(\phi) = \frac{12}{L^2}\,, \quad \dot{\phi} = 0 \Longrightarrow b(z) = \frac{L}{z}\,, \quad f(z) = 1 - \left(\frac{z}{z_h}\right)^4\,, \quad T = \frac{1}{\pi z_h}$$

The 5D Einstein-dilaton model

$$b^2(z) = e^{-\frac{4}{3}\phi(z)}\frac{L^2}{z^2}h(z)\,, \quad \text{Input}: \quad h(z) = \frac{\log(\epsilon)}{\log(\epsilon+(\Lambda z)^2)} \quad \text{Pirner\&Galow'09.}$$





Issues

Thermodynamics

Postulate: Entropy of gauge theories is equal to the Bekenstein-Hawking entropy of their string duals.

$$S(T) = \frac{\mathcal{A}(z_h)}{4G_5} = \frac{V_3 b^3(z_h)}{4G_5} = V_3 s_0 \frac{h^{\frac{3}{2}}(z_h)}{z_h^3}\,, \qquad z_h = \frac{1}{\pi T}$$

High temperature limit: $s(T) \underset{T\to\infty}{\sim} s_0 \pi^3 T^3 = \frac{32}{45}\pi^2 T^3 =: s_{\mathrm{ideal}}(T)$

One can compute all the thermodynamics quantities:

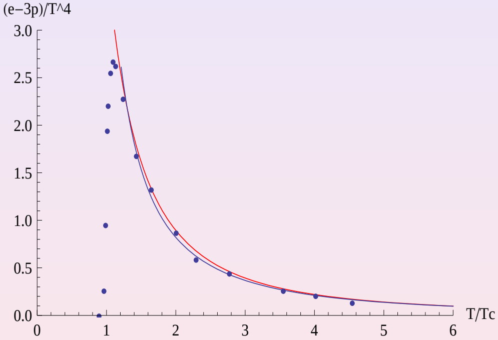
$$s(T) = \frac{d}{dT}p(T)\,, \qquad \Delta(T) \equiv \frac{\epsilon - 3p}{T^4} = \frac{s}{T^3} - \frac{4p}{T^4}\,.$$

One can choose several warp factors:

- Andreev & Zakharov '07: $h_A(z) = e^{1/2cz^2}$.
- Pirner & Galow '09: $h_P(z) = \frac{\log(\epsilon)}{\log(\epsilon + (\Lambda z)^2)}\,, \qquad \epsilon = \frac{l_s^2}{L^2}$.

Thermodynamics

No dilaton dynamics $\Longrightarrow \phi = \mathrm{cte}$:

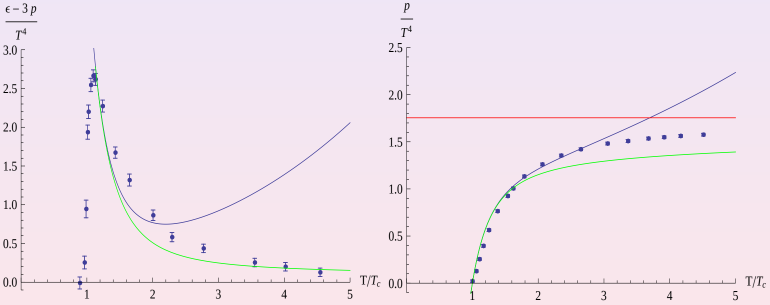


$$1.13 \leq T/T_c \leq 4.5$$

$$\Lambda = 264\mathrm{MeV}\,, \;\; T_c = 270\mathrm{MeV} \Longrightarrow \epsilon = 0.77\,, \;\; \chi^2/\mathrm{dof} = 0.56\,.$$

Thermodynamics

Including dilaton dynamics:



Issues

Polyakov Loop

$$L(T) := \langle P \rangle = \int DX e^{-S_w} \underset{\text{semiclassically}}{\Longrightarrow} \langle P \rangle = \sum_i w_i e^{-S_i} \simeq w_0 e^{-S_0}$$

Nambu-Goto Action:

$$S_{\rm NG} = \frac{1}{2\pi l_s^2} \int d\sigma d\tau \sqrt{\det h_{ab}} = \frac{1}{2\pi l_s^2} \int d\sigma d\tau \sqrt{\det g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu} \,,$$

$$\mu, \nu = t, \vec{x}, z \qquad a, b = \sigma, \tau$$

Modified AdS$_5$-metric at finite temperature:

$$ds_E^2 = \frac{L^2}{z^2} h(z) \left(f(z) dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right) , \qquad f(z) = 1 - \left(\frac{z}{z_h} \right)^4$$

Static configurations: $\tau = t \,, \quad \sigma = z$

$$h_{ab} = \frac{L^2}{z^2} h(z) \begin{pmatrix} f & 0 \\ 0 & \frac{1}{f} + \dot{x}^2 \end{pmatrix} .$$

Polyakov Loop

The NG action writes:

$$S_{\rm NG} = \frac{L^2}{2\pi l_s^2} \int_0^{1/T} d\tau \int_0^{z_h} dz \frac{h(z)}{z^2} \sqrt{1 + \dot{x}^2 f(z)} \,,$$

Equation of motion for x:

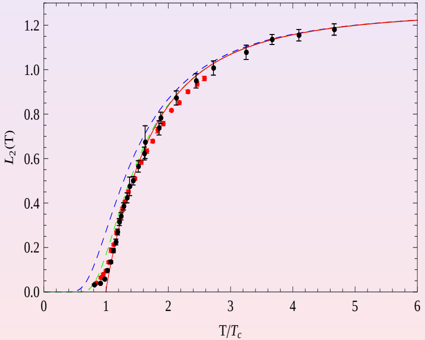
$$\frac{\partial}{\partial z} \left[\frac{h(z)}{z^2} \frac{\dot{x} f}{\sqrt{1+\dot{x}^2 f}} \right] = 0 \,, \qquad \text{Boundary cond.:} \;\; x(0) = x(z_h) \equiv x_0$$

Solution: $\quad x = x_0 =$ **constant.**

$$S_{\rm NG}^{\rm reg} = -\frac{1}{2\epsilon} + \frac{1}{2\pi\epsilon T} \int_0^{z_h} dz \frac{h(z) - 1}{z^2} \,.$$

Polyakov Loop

$$L(T) \simeq w_0 e^{-S_{\mathrm{reg}} - c_R}$$



	$h_A(z)$	$h_P(z)$
ϵ	0.859	0.48
Λ	–	264 MeV
c	1.79 GeV^2	–
T_c	270 MeV	120 MeV

Lattice data ($N_f = 3$): P. Petreczky and K. Petrov, PRD70 (2004), M. Cheng et al., PRD77(2008).

Why are warp factors so good?

$h_A(z)$ and $h_P(z)$ both describe very well lattice results. This is because they include power corrections:

$$h_P(z) = \frac{\log(\epsilon)}{\log\left(\epsilon + (\Lambda z)^2\right)} = 1 + \frac{\Lambda^2}{\epsilon|\log\epsilon|} z^2 + \frac{(2+\log\epsilon)\Lambda^4}{2\epsilon^2 \log^2\epsilon} z^4 + \mathcal{O}(z^6)\,.$$

They are related to condensates, starting from dimension 2.

- Dim 2 condensate: $\mathcal{C}_2 \equiv g^2\langle A_{0,a}^2\rangle = \frac{2N_c}{\pi^2}\frac{\Lambda^2}{\epsilon^2|\log\epsilon|} = (0.50\,\mathrm{GeV})^2$,
- Polyakov loop:

$$L \simeq e^{-S_0} = \exp\left(c_0 - \frac{\mathcal{C}_2}{4N_c T^2} - \frac{\mathcal{C}_4}{16N_c^2 T^4} + \cdots\right),$$

- Trace anomaly:

$$\frac{\epsilon - 3p}{T^4} = \frac{33}{4\pi}\alpha_s \frac{\mathcal{C}_2}{T^2} + \mathcal{O}\left(\frac{\mathcal{C}_4}{T^4}\right)$$

Perturbation theory

But they don't fulfill UV behaviour given by perturbation theory.

$$L = 1 + \underbrace{\frac{4}{3}\sqrt{\pi}\alpha_s^{3/2}}_{\sim 1/\log^{3/2}(T/\Lambda)} + \underbrace{(2\log\alpha_s + 3 + 2\log\pi)\,\alpha_s^2}_{\sim 1/\log^2(T/\Lambda)} + \mathcal{O}(\alpha_s^{5/2}) \quad \text{Gava '81}$$

$$\frac{\epsilon - 3p}{T^4} = \underbrace{\frac{11}{3}\alpha_s^2}_{\sim 1/\log^2(T/\Lambda)} + \mathcal{O}(\alpha_s^{5/2})$$

One can try to construct the corresponding $h(z)$:

$$h(z) \sim \log^{-n}(\Lambda z)\,, \qquad n > 0$$

For the Polyakov loop:

$$h(z) = 1 + \epsilon\frac{16\sqrt{2}}{33\sqrt{11}}\pi^2\left(\frac{1}{\log^{3/2}(1/(\Lambda z))} - \frac{3}{2}\frac{1}{\log^{5/2}(1/(\Lambda z))}\right) \xrightarrow[z\to 0]{} 1$$

A different approximation

Different approximation: choose the form of the dilaton potential (Kiritsis '09, Kajantie '09, Megias '09).

$$\beta(\alpha) = -\beta_0\alpha^2 - \beta_1\alpha^3 + \cdots \Longrightarrow$$
$$\Longrightarrow V(\alpha = e^\phi) = \frac{12}{L^2}\left(1 + \frac{8}{9}\beta_0\alpha + \left(\frac{23}{81} + \frac{4\beta_1^2}{9\beta_0^2}\right)(\beta_0\alpha)^2 + \cdots\right)$$

- Kajantie et al. arXiv:0905.2032 (2009), spatial string tension:

$$V(r) = \sigma_s r\,, \quad \sigma_s(T) = \frac{1}{2\pi\alpha'} b^2(z_h)\alpha^{4/3}(z_h)$$

  This is in contradiction with QCD: $\sigma_{\rm QCD} \sim T^2\alpha_s^2(T)$.
- Megías et al. (2010) compute the Polyakov loop:

$$L(T) = \exp\left(c_0 + \frac{C_*}{2\epsilon}\alpha^{\frac{4}{3}}(z_h) + \mathcal{O}(\alpha^{\frac{7}{3}})\right)$$

  In contradiction with PT: $L_{\rm PT}(T) = \exp\left(\frac{4}{3}\sqrt{\pi}\alpha^{\frac{3}{2}} + \mathcal{O}(\alpha^2)\right)$.

Issues

Heavy $\bar{q}q$ free energy at $T > T_c$

$$e^{-\beta F_{q\bar{q}}} = \left\langle \Omega\left(x = \frac{d}{2}\right) \Omega^\dagger\left(x = -\frac{d}{2}\right) \right\rangle \simeq e^{-S_{\rm NG}}$$

The Nambu-Goto action writes:

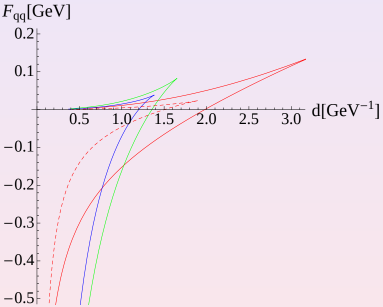
$$S_{\rm NG} = \frac{1}{2\pi\epsilon} \int_0^\beta d\tau \int_{-d/2}^{d/2} dx \frac{h(z)}{z^2} \sqrt{f(z) + \dot{z}^2}\,, \quad X^\mu(\tau, x) = (\tau, x, 0, 0, z(x))\,.$$

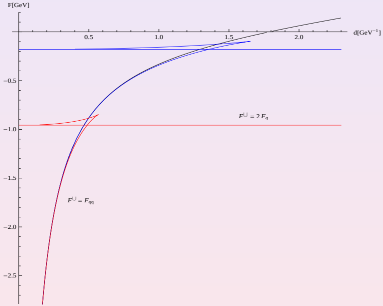
and the heavy $\bar{q}q$ free energy:

$$F_{\bar{q}q} = T \cdot S_{\rm NG}^{\rm reg} = \frac{1}{\pi\epsilon} \left[\int_0^{z_0} \frac{dz}{z^2} \left(\frac{h^2(z) f(z)}{\sqrt{h^2(z) f^2(z) - f(z)\frac{z^4}{C^4}}} \right) - \int_0^{z_h} dz \frac{h(z)}{z^2} \right]$$

$$d = 2 \int_0^{z_0} dz \frac{z^2}{\sqrt{C^4 h^2(z) f^2(z) - f(z) z^4}}\,, \qquad z_0 = z(x = 0)$$

Heavy $\bar{q}q$ free energy at $T > T_c$

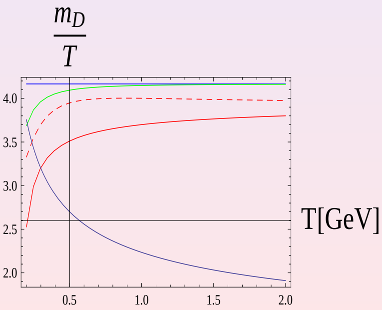




Heavy $\bar{q}q$ free energy at $T > T_c$

Debye mass:

$$F_{\bar{q}q}(d) \sim \frac{e^{-m_D \dot{d}}}{d} \Longrightarrow m_D \sim \frac{1}{d_{\text{screening}}}$$



$$\left.\frac{m_D}{T}\right|_{\text{PT}} \simeq g(T) \cdot \sqrt{\frac{N_c}{3} + \frac{N_f}{6}}$$

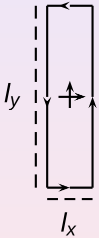
Issues

Spatial Wilson Loops

Rectangular Wilson loop in (x,y) plane:

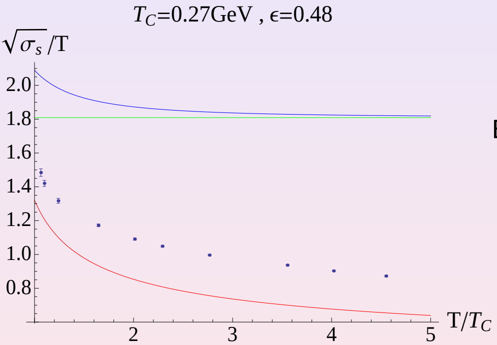
$$\langle W(\mathcal{C})\rangle = \left\langle \exp\left(ig \int_{\mathcal{C}} A_\mu dx^\mu \right) \right\rangle \overset{l_y \to \infty}{\simeq} e^{-l_y \cdot V(d)} ,$$

$$V(d) \overset{d \to \infty}{\simeq} \sigma_s \cdot d$$



$$S_{\mathrm{NG}} = \frac{L^2}{2\pi l_s^2} l_y \int_{-l_x/2}^{l_x/2} dx \frac{h(z)}{z^2} \sqrt{1 + \frac{(\partial_x z)^2}{f(z)}} , \qquad X(x,y) = (x, y, 0, 0, z(x))$$

Spatial Wilson Loops



B.Lucini& M.Teper '02

$$\frac{\sqrt{\sigma_s}}{T} = 0.553\, g^2(T)$$

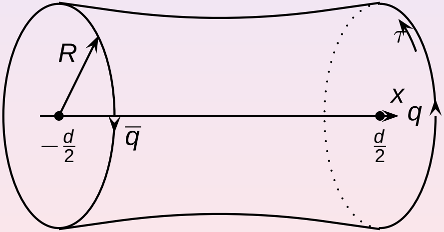
$$\sigma_s(T) = \lim_{d\to\infty} \frac{V(d)}{d} = \frac{1}{2\pi\epsilon}\frac{h(z_h)}{z_h^2} \sim \alpha^{\frac{4}{3}}(T)T^2$$

Issues

Heavy $\bar{q}q$ free energy at $T \leq T_c$

K.Veschgini, E.Megías, J.Nian & H.J.Pirner, arXiv:0911.1680 ('09).

$$e^{-\beta F_{q\bar{q}}(\vec{d},T)} = \frac{1}{N_c^2}\left\langle \mathrm{tr}_c\Omega(\frac{\vec{d}}{2})\,\mathrm{tr}_c\Omega^\dagger(-\frac{\vec{d}}{2})\right\rangle \approx e^{-S_{\rm NG}} \,.$$



$$ds^2 = \frac{h(z)L^2}{z^2}(r^2 d\phi^2 + dr^2 + dx^2 + dz^2)\,, \qquad R = \frac{1}{2\pi T}$$

Heavy $\bar{q}q$ free energy at $T \leq T_c$

$$S_{\mathrm{NG}} = \frac{1}{2\pi l_s^2} \int_0^{2\pi} d\phi \int_{-d/2}^{d/2} dx \, \frac{L^2 h(z)}{z^2} r \sqrt{1 + (z')^2 + (r')^2}$$

- Euler-Lagrange equations:

$$\text{(a)} \qquad k = \frac{h(z) \cdot r}{z^2} \frac{1}{\sqrt{1 + (z')^2 + (r')^2}}$$

$$\text{(b)} \qquad r'' - \frac{h^2(z) \cdot r}{k^2 z^4} = 0$$

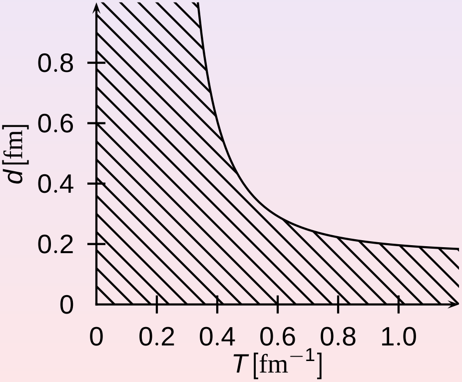
$$\text{(c)} \qquad z'' - \frac{h(z) \cdot r^2 \cdot (z\partial_z h(z) - 2h(z))}{k^2 z^5} = 0$$

- Boundary conditions:

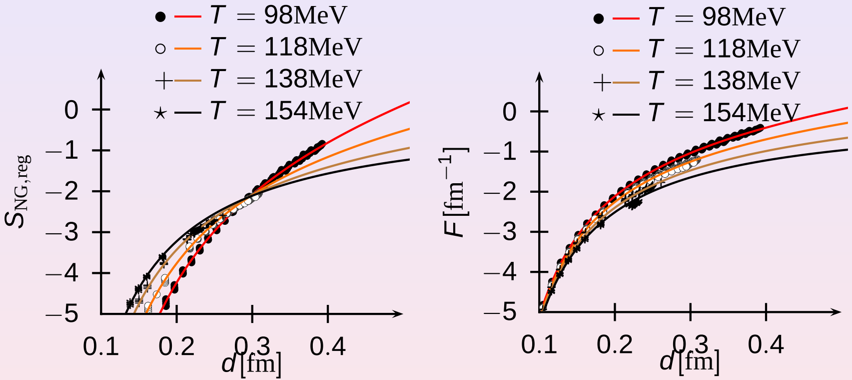
$$r(\pm d/2) = R = \frac{\beta}{2\pi} = \frac{1}{2\pi T}, \qquad z(\pm d/2) = 0 \,.$$

Heavy $\bar{q}q$ free energy at $T \leq T_c$

Numerical solutions only in a limited range.
No minimal surface $\Longrightarrow$ classical approximation not valid.



Heavy $\bar{q}q$ free energy at $T \leq T_c$



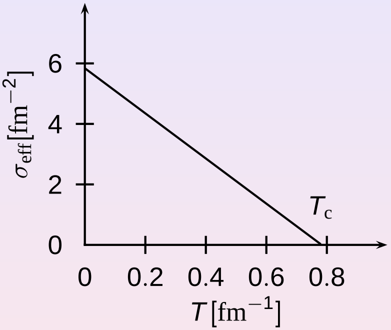
$$F_{\bar{q}q} = T \cdot S^{\rm fit}_{\rm NG,reg} = \frac{-0.48}{d} + d\underbrace{\left(\frac{-7.46}{\rm fm}T + \frac{5.84}{{\rm fm}^2}\right)}_{\equiv \sigma_{\rm eff}(T)} .$$

Heavy $\bar{q}q$ free energy at $T \leq T_c$

- Effective string tension:

$$\sigma_{\rm eff}(T_c) = 0 \Longrightarrow T_c = 154\,\mathrm{MeV}\,.$$

$$T_c^{\mathrm{lattice},N_c=3,N_f=3} \stackrel{\mathrm{Yagi'05}}{=} 155(10)\,\mathrm{MeV}$$



Other thermodynamics quantities:

- Entropy:

$$S_{\bar{q}q} = -\frac{\partial F_{\bar{q}q}}{\partial T} = \frac{7.46}{\mathrm{fm}} d\,,$$

- Inner energy:

$$E_{\bar{q}q} = F_{\bar{q}q} + T\cdot S_{\bar{q}q} = \frac{-0.48}{d} + \frac{5.84}{\mathrm{fm}^2} d\,,$$

Conclusions:

- The non-perturbative behaviour of QCD near and above T_c is characterized by power corrections in T. These power corrections are high energy trace of non-perturbative low energy effects.
- AdS-QCD serves as a powerful tool to study the non perturbative regime of QCD at zero temperature (large distances) and finite temperature (close to the phase transition). We consider conformal breaking warp factors which naturally describe these power corrections.
- We describe at the same time the equation of state of QCD, and the heavy $\bar{q}q$ free energy.