Old and new lessons from the Gross-Neveu model

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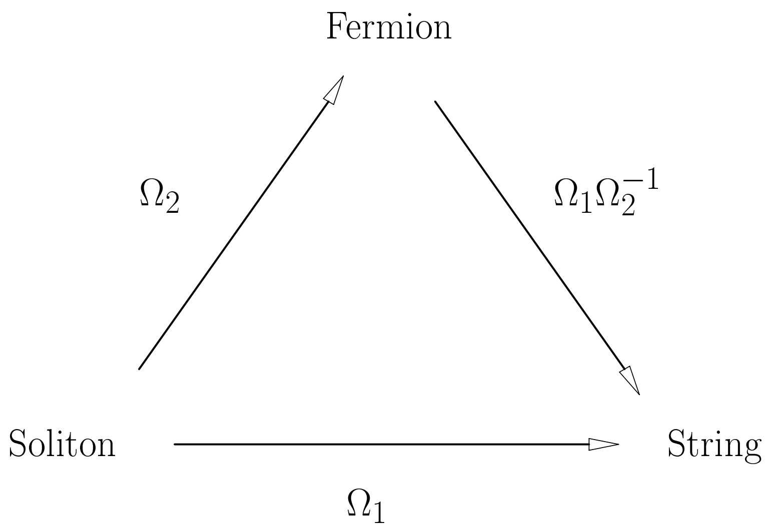
Plan



1) Introduction

$$\mathcal{L} = \sum_{k=1}^{N} \bar{\psi}_k i\partial\!\!\!/ \psi_k + \frac{g^2}{2}\left(\sum_{k=1}^{N} \bar{\psi}_k \psi_k\right)^2$$

Gross, Neveu (1974)

- renormalizable in 1+1 dimensions $\left[g^2\right] = 1$
- 't Hooft limit $N \to \infty,\ Ng^2 = \text{const.}$
- asymptotic freedom
- discrete chiral symmetry $\psi \to \gamma_5 \psi, \quad \bar{\psi}\psi \to -\bar{\psi}\psi$
- dynamical fermion mass m, no confinement

- marginally bound scalar meson with $M_\sigma = 2m$

- solitonic baryons (kink, kink-antikink)

- non-trivial phase diagram as a function of (μ, T)
three phases (massive and massless Fermi gas, kink crystal)
meeting at a tricritical point

- applications to quasi-one-dimensional condensed matter systems
(conducting polymers, carbon nanotubes, superconductors)

- various generalizations

Principal tool in the large N limit: Semi-classical methods –
relativistic version of time-dependent Hartree-Fock (TDHF)
also applicable at finite temperature and chemical potential

Basic mathematical problem

$$\text{TDHF} \qquad (i\partial\!\!\!/ - S)\,\psi_\alpha = 0, \qquad S = -g^2 \sum_{\alpha}^{\text{occ}} \bar{\psi}_\alpha \psi_\alpha$$

Dirac sea $\rightarrow$ Infinite system of coupled, non-linear PDEs

- vacuum: homogeneous condensate – trivial

$$S = m = \Lambda \exp\left(-\frac{\pi}{Ng^2}\right)$$

Dimensional transmutation

- baryons, dense matter: inhomogeneous condensates – non-trivial

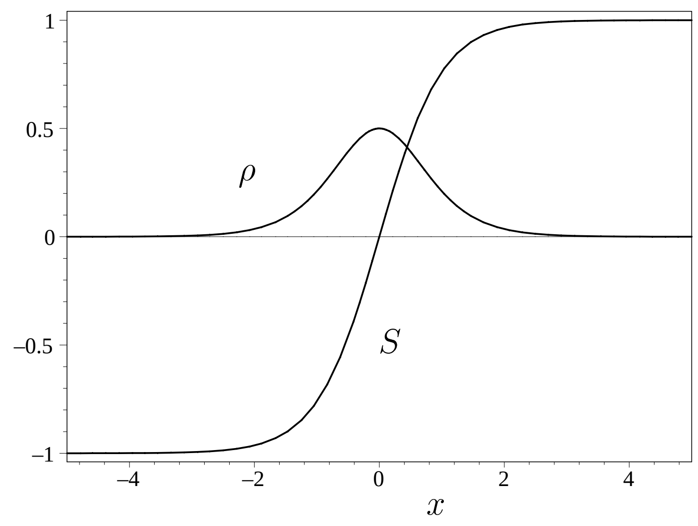
- mesons: quantized fluctuations of Hartree-Fock solution
(relativistic RPA)

Surprisingly, exact analytical solutions can be found in all cases

2) Kink at rest

$S = m \tanh mx \to \tanh x$

(Callan, Coleman, Gross, Zee)



Dirac equation $\to$ effective Schrödinger equation with $1/\cosh^2$ potential (reflectionless). Set $m = 1$ from now on.

Contribution to condensate from continuum- and discrete states

$$\bar{\psi}_k \psi_k = -\frac{1}{\sqrt{k^2+1}} \tanh x, \qquad \bar{\psi}_0 \psi_0 = 0$$

Key to solution of self-consistency problem:

All occupied states give contribution $\sim S$

Example of "type I" solution of TDHF equation

All known analytical solutions of the Gross-Neveu model are type I or type II – only one or two functions enter in $\bar{\psi}_\alpha \psi_\alpha$

In this talk we focus on type I solutions.

Kink has non-trivial fermion number (n valence fermions)

$$N_f = n - \frac{N}{2} = -\frac{N}{2} \dots \frac{N}{2}$$

Related to topology (Jackiw, Rebbi)

Kink mass independent of N_f

$$M_K = \frac{N}{\pi}$$

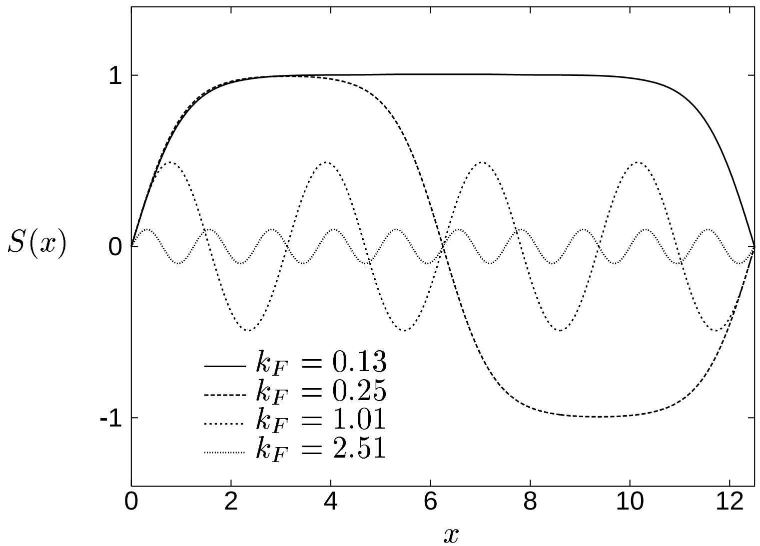
3) Kink crystal

Ground state at finite density: kink crystal favored due to Peierls effect

$$S = \kappa \frac{\mathrm{sn}(x/\kappa,\kappa)\mathrm{cn}(x/\kappa,\kappa)}{\mathrm{dn}(x/\kappa,\kappa)}$$

$$\rho = \frac{1}{2\kappa \mathrm{K}}$$

Lamé equation – finite band potential



type I solution

$$\bar{\psi}_\alpha \psi_\alpha = -\frac{\kappa\, \mathrm{dn}(\alpha,\kappa)}{\mathrm{dn}^2(\alpha,\kappa) - \mathrm{E}/\mathrm{K}} S$$

At finite temperature, same functional form of S, but type II solution.

4) Boosted kink and structure functions

Lorentz boost

$$S(x) \to S(\gamma(x - vt)), \qquad \gamma = (1 - v^2)^{-1/2}$$

TDHF gives covariant energy-momentum relation for the kink baryon

$$E = \gamma M_K, \qquad P = \gamma v M_K$$

Fermion (quark) and antifermion (antiquark) momentum distribution in arbitrary frame

$$\begin{aligned} W_q(k) &= \langle HF | a_k^\dagger a_k | HF \rangle \\ W_{\bar{q}}(k) &= 1 - \langle HF | b_{-k}^\dagger b_{-k} | HF \rangle \end{aligned}$$

Free, massive fermion operators a_k, b_k . Rescaling

$$k = x \hat{P}_B = x \frac{P_B}{N}, \qquad w_{q,\bar{q}}(x) = \hat{P}_B W_{q,\bar{q}}(x \hat{P}_B)$$

Infinite momentum frame

$$x \in [0, N] \to [0, \infty]$$

Kink structure functions (fully occupied valence level)

$$\begin{aligned} w_{\bar{q}}(x) &= \int_0^\infty \mathrm{d}q \frac{2}{(\pi^2 + 4q^2)\sinh^2(x+q)} \\ w_q(x) &= \frac{1}{\cosh^2 x} + w_{\bar{q}}(x) \end{aligned}$$

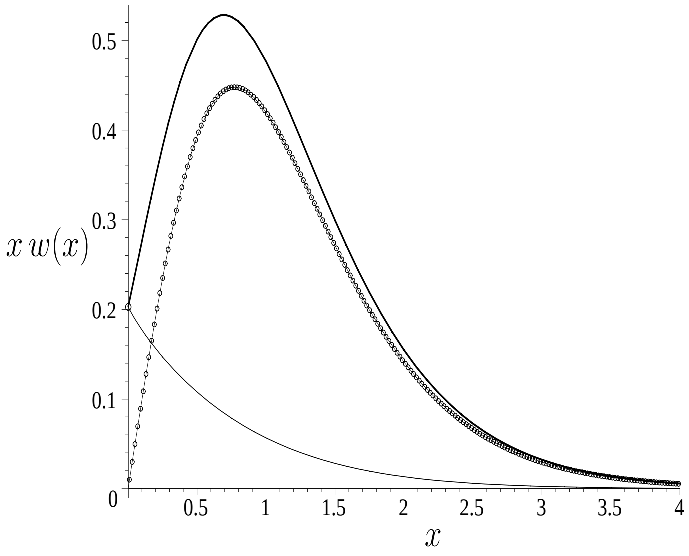
Sum rules

$$\begin{aligned} 1 &= \int_0^\infty \mathrm{d}x \left(w_q(x) - w_{\bar{q}}(x) \right) \\ 1 &= \int_0^\infty \mathrm{d}x\, x \left(w_q(x) + w_{\bar{q}}(x) \right) \end{aligned}$$

Logarithmic infrared divergence in the number of quarks and antiquarks from low x region

$$w_q(x) \approx w_{\bar{q}}(x) \approx \frac{2}{\pi^2 x} \qquad (x \to 0)$$

Momentum distribution $x w_{q,\bar{q}}(x)$ well behaved



Quark (thick line), antiquark (thin line) and valence quark (dotted line) structure functions for the kink baryon

Infinite momentum frame: Fraction of baryon momentum carried by

- valence quarks: $\frac{\ln 2}{2} = 35\%$
- sea quarks: $\frac{1}{2} = 50\%$
- antiquarks: $\frac{1-\ln 2}{2} = 15\%$

5) Kink-antikink scattering

Dashen, Hasslacher, Neveu (1975): kink-antikink breather, guessed by analogy with sine-Gordon breather

According to our classification: type II solution of TDHF

Analytic continuation $\epsilon \to i/v$: Kink-antikink scattering

Result actually simpler than the breather due to different boundary conditions in scattering problem

Self-consistent potential

$$S = \frac{v \cosh 2\gamma x - \cosh 2\gamma v t}{v \cosh 2\gamma x + \cosh 2\gamma v t}$$

describes repulsive kink-antikink encounter with asymptotic velocities $\pm v$ in cm frame

Properties of the solution

- continuum states $\sim \mathrm{e}^{i(kx-\omega t)}, \omega = -\sqrt{k^2+1}$ (reflectionless)

- two bound states do not contribute to condensate, but lead to exchange of valence fermions between kink and antikink

- type I solution

$$\bar{\psi}_k \psi_k = -\frac{1}{\sqrt{k^2+1}} S, \qquad \bar{\psi}_0 \psi_0 = 0$$

- time delay

$$\Delta t = \frac{\ln v}{v}\sqrt{1-v^2} < 0$$

- kink and antikink interchange fermion number during the collision

$$K(n-N/2) + \bar{K}(\bar{n}-N/2) \longrightarrow K(\bar{n}-N/2) + \bar{K}(n-N/2)$$

$\rightarrow$ Maple animation

6) Effective bosonic theory

Static solutions: Non-linear Schrödinger equation (ϕ^4 theory)

$$S'' - 2S^3 + cS = 0$$

- c depends on particular solution

- kink-antikink scattering fails (no solitons in ϕ^4 theory)

For type I solutions: TDHF equivalent to $N = 1$ classical Gross-Neveu model

Neveu and Papanicolaou (1978): Proof of integrability for $N = 1, 2$

TDHF equation

$$(i\partial\!\!\!/ - S)\psi = 0$$

Dirac matrices

$$\gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \quad \gamma_5 = -\sigma_3$$

Light-cone coordinates

$$z = x - t, \quad \bar{z} = x + t$$

Dirac equation

$$\begin{aligned} -2i\psi_{1,z} &= S\psi_2 \\ 2i\psi_{2,\bar{z}} &= S\psi_1 \end{aligned}$$

Self-consistency for type I solutions

$$S = \ell\bar{\psi}\psi = \ell\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right)$$

What about the other derivatives $\psi_{1,\bar{z}}, \psi_{2,z}$? Identities

$$\begin{aligned} S\psi_{1,\bar{z}} - S_{,\bar{z}}\psi_1 &= -ih_1\ell\psi_2 \\ S\psi_{2,z} - S_{,z}\psi_2 &= -ih_2\ell\psi_1 \end{aligned}$$

Have introduced

$$\begin{aligned} h_1 &= i\left(\psi_1^*\psi_{1,\bar{z}} - \psi_1\psi_{1,\bar{z}}^*\right) \\ h_2 &= i\left(\psi_2^*\psi_{2,z} - \psi_2\psi_{2,z}^*\right) \end{aligned}$$

with

$$h_{1,z} = 0 = h_{2,\bar{z}} \quad \longrightarrow \quad h_{1,2} = \text{const.}$$

Express all derivatives of $\psi_{1,2}$ through $\psi_{1,2}$

$$\psi_{,\bar{z}} = C_1\psi, \quad \psi_{,z} = C_2\psi$$

with

$$C_1 = \begin{pmatrix} S_{,\bar{z}}S^{-1} & -ih_1\ell S^{-1} \\ -iS/2 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & iS/2 \\ -ih_2\ell S^{-1} & S_{,z}S^{-1} \end{pmatrix}$$

Integrability condition

$$C_{1,z} - C_{2,\bar{z}} + [C_1, C_2] = 0$$

reminiscent of non-Abelian field strength tensor

Equivalently

$$SS_{,z\bar{z}} - S_{,z}S_{,\bar{z}} - \frac{1}{4}S^4 = h_1 h_2 \ell^2 = -\frac{1}{4}$$

Change of variables

$$S = \mathrm{e}^{\theta/2} \quad \longrightarrow \quad \theta_{z,\bar{z}} = \sinh\theta$$

C_1, C_2 Lax pair of sinh-Gordon equation

Discussion

• Kink, kink-crystal and kink-antikink satisfy classical sinh-Gordon equation after the singular transformation $S = \mathrm{e}^{\theta/2}$ or $\theta = \ln S^2$

• Linearized equation in ordinary coordinates

$$\partial_\mu \partial^\mu \theta + 4\sinh\theta \approx (\partial_\mu \partial^\mu + 4)\,\theta = 0$$

yields Klein-Gordon equation for scalar meson with mass 2

• Relation to ϕ^4 theory for static case? Functional separable solution of sinh-Gordon equation

$$\theta(x,t) = 4\,\mathrm{artanh}\,[f(t)g(x)]$$

$$f_{,t}^2 = Af^4 + Bf^2 + C, \quad -g_{,x}^2 = Cg^4 + (B+4)g^2 + A$$

• General N soliton solution known for sinh-Gordon equation – candidate for TDHF solution of the Gross-Neveu model with N kinks and antikinks

• Soliton theory provides us with self-consistent potential and TDHF wave functions if Lax pair is known

• Relation between kink, σ meson and sinh-Gordon equation in massless Gross-Neveu model analogous to baryon, π meson and sine-Gordon equation in massive NJL$_2$ model close to chiral limit (derivative expansion). Picture of baryons reminiscent of Skyrme model.

7) Relation to string theory

Strings in AdS_3 related to sinh-Gordon equation (Jevicki, Jin et al 2007-2010). Start from Lax pair of σ model (Pohlmeyer reduction)

$$\begin{aligned} \phi_{,\bar{z}} &= A_1\phi, \quad \phi_{,z} = A_2\phi \\ \psi_{,\bar{z}} &= B_1\psi, \quad \psi_{,z} = B_2\psi \end{aligned}$$

Integrability condition for both pairs of equations

$$\alpha_{,z\bar{z}} = \sinh\alpha$$

Normalization,

$$1 = \phi_1^*\phi_1 - \phi_2^*\phi_2 = \psi_1^*\psi_1 - \psi_2^*\psi_2$$

String coordinates

$$\begin{aligned} Z_1 &= Y_{-1} + iY_0 = \phi_1^*\psi_1 - \phi_2^*\psi_2 \\ Z_2 &= Y_1 \;\; + iY_2 = \phi_2^*\psi_1^* - \phi_1^*\psi_2^* \end{aligned}$$

Satisfy Virasoro constraints and string equation in conformal gauge.

Every solution of sinh-Gordon equation can be transformed into a classical solution of string rotating in AdS_3; solitons $\cong$ spikes. $\rightarrow$ Examples

Relationship between Gross-Neveu model and string theory

- type I solutions of TDHF

$$\lim_{N\to\infty} \text{ quantum Gross}-\text{Neveu} \leftrightarrow N = 1 \text{ classical Gross}-\text{Neveu}$$

- Neveu and Papanicolaou

$$N = 1 \text{ classical Gross}-\text{Neveu} \leftrightarrow \text{ classical sinh}-\text{Gordon}$$

- Jevicki et al

$$\text{classical sinh}-\text{Gordon} \leftrightarrow \text{ classical strings in AdS3}$$

Relate quantum Gross-Neveu model in the large N limit to classical strings in AdS_3. Basic structural element

$$C_{1,z} - C_{2,\bar{z}} + [C_1, C_2] = 0$$

Similar to (vanishing) non-Abelian field strength tensor – " pure gauge". Therefore both mappings involve non-Abelian "gauge transformations".

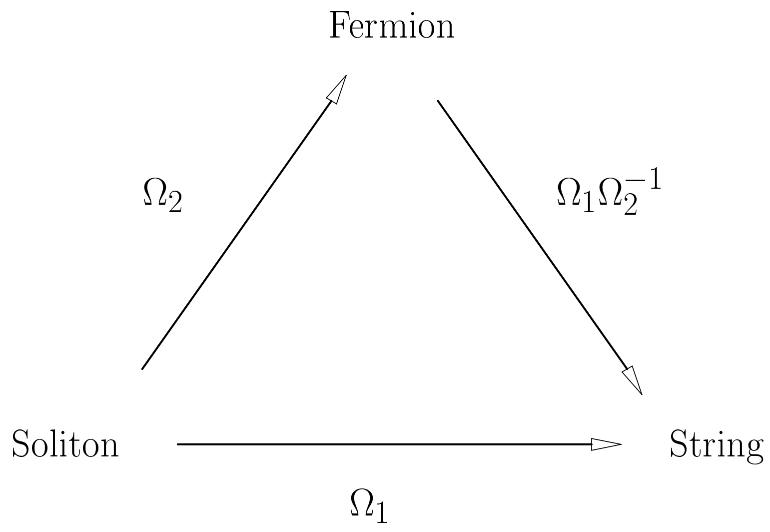
Jevicki et al find the gauge transformation from the sinh-Gordon model to the string-related σ model

$$A_1 = \Omega_1(U - \bar{\partial})\Omega_1^{-1}, \quad A_2 = \Omega_1(V - \partial)\Omega_1^{-1}$$

Neveu and Papanicolaou find the gauge transformation from the sinh-Gordon model to the Gross-Neveu model

$$C_1 = \Omega_2(U - \bar{\partial})\Omega_2^{-1}, \quad C_2 = \Omega_2(V - \partial)\Omega_2^{-1}$$

Type I solutions of the large N Gross-Neveu model can therefore be mapped to string theory by the gauge transformation $\Omega = \Omega_1\Omega_2^{-1}$



Ω_1 and Ω_2 are fairly complicated. How bad is $\Omega = \Omega_1 \Omega_2^{-1}$?

Need three cases $\zeta = -\lambda/2, \pm i\lambda/2$ where $\zeta = (k-\omega)/2$ is the spectral parameter of the TDHF fermion spinor

Simplified notation: $\vartheta = \theta(z,\bar{z})/4, \;\; \eta = \lambda^{-1}$

$$\Omega_1(\text{Jevicki}) \qquad \Omega_2^{-1}(\text{Neveu}) \quad = \quad \Omega$$

$$\frac{\sqrt{i}}{2}\begin{pmatrix} e^{\vartheta} - e^{-\vartheta} & -e^{\vartheta} - e^{-\vartheta} \\ -ie^{\vartheta} - ie^{-\vartheta} & ie^{\vartheta} - ie^{-\vartheta} \end{pmatrix}\begin{pmatrix} -\eta & e^{-2\vartheta} \\ -\eta & -e^{-2\vartheta} \end{pmatrix} = \sqrt{i}e^{-\vartheta}\begin{pmatrix} \eta & 1 \\ i\eta & -i \end{pmatrix}$$

$$-\frac{1}{2}\begin{pmatrix} ie^{\vartheta} - e^{-\vartheta} & -ie^{\vartheta} - e^{-\vartheta} \\ e^{\vartheta} - ie^{-\vartheta} & -e^{\vartheta} - ie^{-\vartheta} \end{pmatrix}\begin{pmatrix} -i\eta & e^{-2\vartheta} \\ -i\eta & -e^{-2\vartheta} \end{pmatrix} = -ie^{-\vartheta}\begin{pmatrix} \eta & 1 \\ i\eta & -i \end{pmatrix}$$

$$-\frac{1}{2}\begin{pmatrix} e^{\vartheta} - ie^{-\vartheta} & -e^{\vartheta} - ie^{-\vartheta} \\ -ie^{\vartheta} + e^{-\vartheta} & ie^{\vartheta} + e^{-\vartheta} \end{pmatrix}\begin{pmatrix} i\eta & e^{-2\vartheta} \\ i\eta & -e^{-2\vartheta} \end{pmatrix} = -e^{-\vartheta}\begin{pmatrix} \eta & 1 \\ i\eta & -i \end{pmatrix}$$

As a result, the construction of AdS$_3$ string solution Z_1, Z_2 from Gross-Neveu TDHF type I solution $\Psi(\zeta)$ and vice versa is very simple.

Define

$$\psi_a = \sqrt{i}\Psi(\zeta = -\lambda/2), \quad \psi_b = \frac{1}{\sqrt{2}}[\Psi(\zeta = -i\lambda/2) + i\Psi(\zeta = i\lambda/2)]$$

Normalization conditions

$$\begin{aligned} 1 &= \phi_1^* \phi_1 - \phi_2^* \phi_2 = \frac{2}{\lambda S}\bar{\psi}_a \psi_a \\ 1 &= \psi_1^* \psi_1 - \psi_2^* \psi_2 = \frac{2}{\lambda S}\bar{\psi}_b \psi_b \end{aligned}$$

String coordinates

$$\begin{aligned} Z_1 &= Y_{-1} + iY_0 = \phi_1^* \psi_1 - \phi_2^* \psi_2 = \frac{2}{\lambda S}\bar{\psi}_a \psi_b = \frac{\bar{\psi}_a \psi_b}{\bar{\psi}_a \psi_a} \\ Z_2 &= Y_1 \quad + iY_2 = \phi_2^* \psi_1^* - \phi_1^* \psi_2^* = -\frac{2}{\lambda S}\bar{\psi}_a i\gamma_5 \psi_b^* = -\frac{\bar{\psi}_a i\gamma_5 \psi_b^*}{\bar{\psi}_a \psi_a} \end{aligned}$$

Example

Vacuum of Gross-Neveu model

$$(i\partial\!\!\!/ - m)\,\Psi = 0, \qquad m = 1$$

Plane wave solution

$$\Psi(\zeta) = \begin{pmatrix} \zeta \\ -1/2 \end{pmatrix} \mathrm{e}^{i(\bar{z}\zeta - z/4\zeta)}$$

Plug it in $\to$ rotating string

$$Z_1 = -\mathrm{e}^{iA_-}\cosh A_+, \quad Z_2 = i\mathrm{e}^{iA_-}\sinh A_+, \quad A_\pm = \frac{\bar{z}\lambda}{2} \pm \frac{z}{2\lambda}$$

AdS_3 embedding

$$-|Z_1|^2 + |Z_2|^2 = -1$$

Conformal gauge equation of motion and Virasoro constraints

$$|Z_{1,z}|^2 - |Z_{2,z}|^2 = 0, \quad |Z_{1,\bar{z}}|^2 - |Z_{2,\bar{z}}|^2 = 0$$

satisfied