A non-equilibrium statistical basis for the relativistic diffusion model

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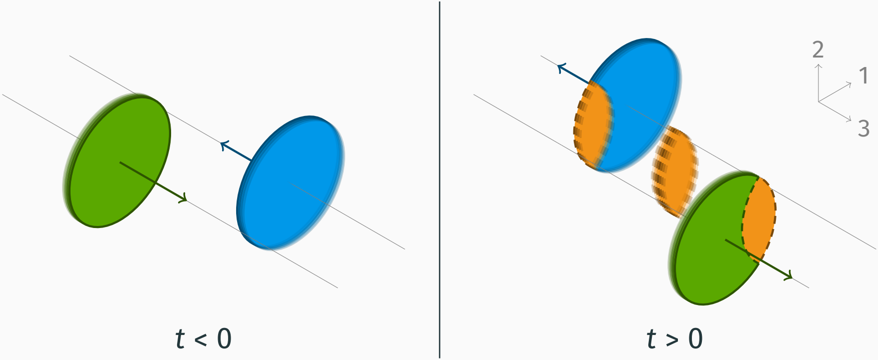
Motivation



Idea: model dynamics of produced particles via diffusion

- heat bath = light partonic d.o.f.
- Brownian particles = charged hadrons, net-protons, ...

Outline

The relativistic diffusion model

Recap: Transverse mass & rapidity

transverse plane

$$m_\perp(p^1, p^2) = \sqrt{m^2 + (p^1)^2 + (p^2)^2}$$

$$(p^1, p^2, p^3) \longmapsto (m_\perp, \cancel{\phi_\perp}, y)$$

beam direction

$$y(p^1, p^2, p^3) = \frac{1}{2}\ln\left[\frac{E(p^1,p^2,p^3)+p^3}{E(p^1,p^2,p^3)-p^3}\right]$$

longitudinal momentum	single particle energy
$\underbrace{p_\parallel(m_\perp, y)}_{\equiv p^3(m_\perp, y)} = m_\perp \sinh(y)$	$E(m_\perp, y) = m_\perp \cosh(y)$

central collisions ⇨ neglect anisotropy in transverse plane

Ingredient 1: Three sources of particle production

rapidity spectrum

number of particles in source

$$\frac{\mathrm{d}N}{\mathrm{d}y}(t;y) = \sum_{k\in\{\mathrm{f1},\mathrm{f2},\mathrm{gg}\}} N_k\,\psi_k(t;y)$$

two fragmentation sources & one central (gluon–gluon) source

probability densities in rapidity space

[G. Wolschin, Phys. Lett. B **569**, 67 (2003)]

Ingredient 2: Time-evolution via Fokker–Planck equation

$$\underbrace{\partial_t R_k}_{\text{time evolution}} = -\underbrace{\partial_y\left[J_k R_k\right]}_{\text{drift}} + \underbrace{\partial_y^2\left[D_k R_k\right]}_{\text{diffusion}}$$

with

$$\psi_k(t;y) =: \int \mathrm{d}m_\perp\, m_\perp\, E(m_\perp, y)\, R_k(t; m_\perp, y)$$ "reduced" distribution

$$J_k(m_\perp, y) = -A_k(m_\perp)\sinh(y)$$ drift coefficient

$$D_k(m_\perp) = A_k(m_\perp)/(\beta m_\perp)$$ diffusion coefficient

[G. Wolschin, Phys. Lett. B **569**, 67 (2003)]

Issues regarding the current RDM

removal of superfluous d.o.f. from FPE?

conceptual

validity of FPE for "reduced" distribution R?

relation between drift ↔ diffusion?

physical

underlying microscopic processes?

Stochastic treatment of diffusion processes

Stochastic process = sequence of $N + 1$ random variables

$$\left\{W_t\right\}_{t \in T} = \left\{W_{t_0} \to W_{t_1} \to W_{t_2} \to \cdots \to W_{t_N}\right\}$$

where each W_{t_i} randomly takes some realization $w_i \in \Omega$

⚠ randomly $\nRightarrow$ independently

Special case: Markov chain

$\mathbb{P}\left\{W_{t_{n+1}} = w_{n+1}\right\}$ depends solely on w_n (i. e. not on $w_0, \ldots, w_{n-1}$)

Wiener process

1-dimensional simple symmetric random walk

$$W_{t_{n+1}} = W_{t_n} \pm \Delta w$$

with equal probability $p = ½$ for going left/right

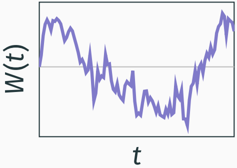
Special choice:

$t_n = \frac{n}{N}$ and $\Delta w = \frac{1}{\sqrt{N}}$

$N \to \infty$

⇨

Wiener process



Behavior of $\mathrm{d}W(t) := W(t + \mathrm{d}t) - W(t)$ for (arbitrary) timestep $\mathrm{d}t$:

$$\langle \mathrm{d}W(t) \rangle = 0 \,, \qquad \langle \mathrm{d}W(t)\,\mathrm{d}W(s) \rangle = \begin{cases} \mathrm{d}t & \text{for } t = s \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}\{\mathrm{d}W(t) \in [w, w + \mathrm{d}w]\} = (2\pi\,\mathrm{d}t)^{-½} \exp\left[-\frac{w^2}{2\,\mathrm{d}t}\right] \mathrm{d}w$$

incompatible with SRT!

Non-relativistic diffusion in one-dimensional Euclidean space

Brownian motion in 1 dimension

$$\mathrm{d}X(t) = \sigma\,\mathrm{d}W(t)$$

$\Downarrow$ define probability density ρ

$$\rho(t;x)\,\mathrm{d}x \coloneqq \mathbb{P}\{X(t) \in [x, x + \mathrm{d}x]\}$$

$\Downarrow$ expand $\langle h[X(t + \mathrm{d}t])\rangle$ for test function h

1-dimensional diffusion equation

$$\partial_t \rho(t;x) = \frac{1}{2}\sigma^2 \partial_x^2 \rho(t;x)$$

[Itô, Fisk, Stratonovich, Klimontovich]

Non-relativistic diffusion in multi-dimensional Euclidean space

Brownian motion in d dimensions

$$\mathrm{d}\vec{X}(t) = \overleftrightarrow{\sigma} \cdot \mathrm{d}\vec{W}(t)$$

$\Downarrow$ define probability density ρ

$$\rho(t;\vec{x})\,\mathrm{d}^d x \coloneqq \mathbb{P}\Big\{X_i(t) \in [x_i, x_i + \mathrm{d}x_i]\ \forall\, i = 1, \dots, d\Big\}$$

$\Downarrow$ expand $\langle h[\vec{X}(t + \mathrm{d}t]]\rangle$ for test function h

d-dimensional diffusion equation

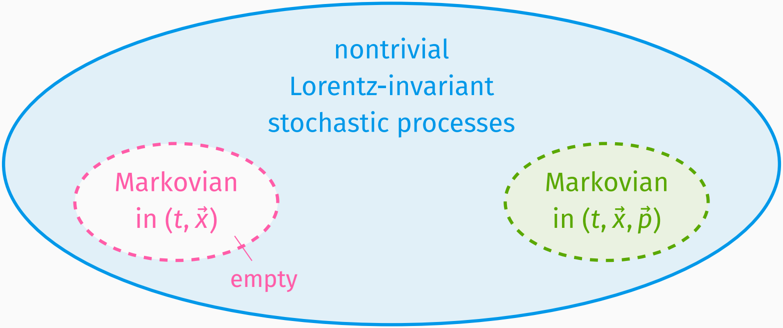
$$\partial_t \rho(t;\vec{x}) = \frac{1}{2}\Big(\overleftrightarrow{\sigma}^{\mathsf{T}} \cdot \vec{\nabla}_x\Big)^2 \rho(t;\vec{x})$$

[Itô, Fisk, Stratonovich, Klimontovich]

Relativistic diffusion in Minkowski spacetime?

Theorem (Dudley, Hakim, Łopuszański)

Nontrivial Lorentz-invariant Markov processes cannot exist in Minkowski spacetime $(t, \vec{x})$!



Relativistic diffusion processes in phase-space

Langevin-dynamical ansatz (I)

Premises

- reference frame = rest frame of background (medium)
- evolution parameter $t \equiv X^0$
- on-shell particles: $E(\vec{P}) = \sqrt{M^2 + \vec{P}^2} \equiv P^0$
- particle–particle interactions negligible

Equations of motion:

$$\mathrm{d}\vec{X} = \frac{\vec{P}}{E(\vec{P})}\,\mathrm{d}t\,, \qquad \mathrm{d}\vec{P} = \underbrace{\vec{\mu}(\vec{P})\,\mathrm{d}t}_{\text{drift}} + \underbrace{\overleftrightarrow{\sigma}(\vec{P})\cdot\mathrm{d}\vec{W}}_{\text{diffusion}}$$

Too general for our purposes!

[Debbasch et al., Dunkel and Hänggi]

Langevin-dynamical ansatz (II)

$$P_\parallel = M_\perp \sinh(Y)$$
$$E = M_\perp \cosh(Y)$$

Simplification steps:

1. exploit symmetry ⇨ switch to $(m_\perp, y)$ ⇨ 2D problem

$$\mathrm{d}X_\perp = \frac{\sqrt{M_\perp^2 - M^2}}{M_\perp \cosh(Y)}\,\mathrm{d}t\,, \qquad \mathrm{d}X_\parallel = \tanh(Y)\,\mathrm{d}t$$

2. assume negligible interdependence of $\mathrm{d}M_\perp$ & $\mathrm{d}Y$

$$\mathrm{d}M_\perp = \mu_\perp(M_\perp)\,\mathrm{d}t + \sigma_\perp(\cancel{M_\perp})\,\mathrm{d}W_\perp$$
$$\mathrm{d}Y = \mu_\parallel(Y)\,\mathrm{d}t + \sigma_\parallel(\cancel{Y})\,\mathrm{d}W_\parallel$$

3. *optional:* assume constant diffusion coefficients

Time-evolution of the distribution function (I)

Phase-space distribution function

$$\Psi(t; x_\perp, x_\parallel, m_\perp, y)\, \mathrm{d}x_\perp\, \mathrm{d}x_\parallel\, \mathrm{d}m_\perp\, \mathrm{d}y := \mathbb{P}\Big\{X_\perp(t) \in [x_\perp, x_\perp + \mathrm{d}x_\perp] \wedge X_\parallel(t) \in [x_\parallel, x_\parallel + \mathrm{d}x_\parallel] \wedge M_\perp(t) \in [m_\perp, m_\perp + \mathrm{d}m_\perp] \wedge Y(t) \in [y, y + \mathrm{d}y]\Big\}$$

probability to find a particle at time t in configuration $(x_\perp, x_\parallel, m_\perp, y)$

Evolution equation:

$$\partial_t \Psi = -\frac{\sqrt{m_\perp^2 - M^2}}{m_\perp \cosh(y)}\, \partial_{x_\perp} \Psi - \tanh(y)\, \partial_{x_\parallel} \Psi - \partial_{m_\perp}\big[\mu_\perp(m_\perp)\,\Psi\big] + \tfrac{1}{2}\sigma_\perp^2 \partial_{m_\perp}^2 \Psi - \partial_y\big[\mu_\parallel(y)\,\Psi\big] + \tfrac{1}{2}\sigma_\parallel^2 \partial_y^2 \Psi$$

still too much information

⇩

integrate out spatial & transverse d.o.f.

Time-evolution of the distribution function (II)

Marginalized (longitudinal) distribution function

$$\psi(t;y) = \int \mathrm{d}x_\perp \, \mathrm{d}x_\parallel \, \mathrm{d}m_\perp \, \Psi(t;x_\perp,x_\parallel,m_\perp,y)$$

Integrated evolution equation:

$$\underbrace{\partial_t\,\psi(t;y)}_{\text{time evolution}} = \underbrace{-\partial_y\big[\mu_\parallel(y)\,\psi(t;y)\big]}_{\text{drift}} + \underbrace{\tfrac{1}{2}\sigma_\parallel^2\partial_y^2\,\psi(t;y)}_{\text{diffusion}}$$

⇨ FPE for "proper" distribution ψ
⇨ only one remaining d.o.f.
} conceptual issues ✓

Fluctuation–dissipation relation

Asymptotic equilibration of the system

Observation: system approaches stationary equilibrium state

- momentum space: isotropic asymptote expected

$$\lim_{t\to\infty} \phi(t;\vec{p}) =: \phi_\infty(\vec{p}) = C_3^{-1} \exp\left[-\beta\, E(\vec{p})\right] \qquad \text{Maxwell–Jüttner distribution}$$

- rapidity space: transform ϕ_∞ to $(m_\perp, \varphi_\perp, y)$ ⇨ integrate out

$$\psi_\infty(y) := \int_M^\infty \mathrm{d}m_\perp \int_0^{2\pi} \mathrm{d}\varphi_\perp \underbrace{m_\perp^2 \cosh(y)}_{\text{Jacobian}} \phi[t;\vec{p}(m_\perp,\varphi_\perp,y)]$$

$$= C_3^{-1} F(y) \exp\left[-\beta M \cosh(y)\right]$$

with

$$F(y) := \frac{2\pi M^2}{\beta}\left\{\frac{2}{[\beta M \cosh(y)]^2} + \frac{2}{\beta M \cosh(y)} + 1\right\}$$

[F. Forndran and G. Wolschin, Eur. Phys. J. A **53**, 37 (2017)]

Connection of drift & diffusion

Consistency: ψ_∞ must be a solution of the evolution equation

$$\underbrace{\partial_t \psi_\infty}_{=0} = \partial_y\left[-\mu_\parallel \psi_\infty + \tfrac{1}{2}\sigma_\parallel^2 \partial_y \psi_\infty\right] \quad \Rightarrow \quad \psi_\infty(y) \propto \exp\left[\int_*^y \mathrm{d}\tilde{y}\, \frac{\mu_\parallel(\tilde{y})}{\sigma_\parallel^2/2}\right]$$

$$\Updownarrow$$

Fluctuation–dissipation relation

$$\mu_\parallel(y) = \tfrac{1}{2}\sigma_\parallel^2 \partial_y \ln\left[\psi_\infty(y)\right] = \underbrace{\tfrac{1}{2}\sigma_\parallel^2\left\{-\beta M \sinh(y)\right.}_{\text{equivalent to RDM}} + \underbrace{\left.\frac{F'(y)}{F(y)}\right\}}_{\text{new term!}}$$

[Klimontovich]

Conclusion & Outlook

Revisiting the issues ...

conceptual

FPE contains
only one d.o.f.!

!

FPE for “proper”
probability density ψ!

physical

revised FDR
more suitable?

?

determine $\sigma_{\parallel}$ via
mean relaxation?

Thank you!

Further reading

A. Einstein, Ann. Phys. **322**, 549 (1905).

M. von Smoluchowski, Ann. Phys. **353**, 1103 (1916).

G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).

K. Itô, Proceedings of the Japan Academy **22**, 32 (1946).

D. L. Fisk, Trans. Amer. Math. Soc. **120**, 369 (1965).

R. L. Stratonovich, SIAM J. Control **4**, 362 (1966).

Y. L. Klimontovich, Physica A **163**, 515 (1990).

F. Debbasch, K. Mallick, and J. P. Rivet, J. Stat. Phys. **88**, 945 (1997).

G. Wolschin, Phys. Lett. B **569**, 67 (2003).

J. Dunkel and P. Hänggi, Phys. Rep. **471**, 1 (2009).

F. Forndran and G. Wolschin, Eur. Phys. J. A **53**, 37 (2017).

P. Schulz and G. Wolschin, Mod. Phys. Lett. A **33**, 1850098 (2018).

B. Kellers and G. Wolschin, Prog. Theor. Exp. Phys. **2019** (2019).

Backup

Evolution in phase-space

Phase-space distribution function

$$f(t; x^1, \dots, x^d, p^1, \dots, p^d)\, \mathrm{d}^d x\, \mathrm{d}^d p$$
$$\coloneqq \mathbb{P}\Big\{(X^i(t), P^i(t)) \in [x^i, x^i + \mathrm{d}x^i] \times [p^i, p^i + \mathrm{d}p^i] \text{ for } i = 1, \dots, d\Big\}$$

$$\left(\partial_t + \frac{p^i}{E}\partial_{x^i}\right) f = \partial_{p^i}\left[-\mu^i_{\text{◐}} f + \tfrac{1}{2}\partial_{p_j}\left(\sigma^i{}_k \sigma_j{}^k f\right)\right] \qquad \text{Itô}$$

$$= \partial_{p^i}\left[-\mu^i_{\text{●}} f + \tfrac{1}{2}\sigma^i{}_k \partial_{p_j}\left(\sigma_j{}^k f\right)\right] \qquad \text{Stratonovich}$$

$$= \partial_{p^i}\left[-\mu^i_{\text{◑}} f + \tfrac{1}{2}\sigma^i{}_k \sigma_j{}^k \partial_{p_j} f\right] \qquad \text{Klimontovich}$$

Counter terms in drift coefficient function

$$\mu^i_{\circledcirc} = \mu^i_{\text{●}} - \tfrac{1}{4}\lambda_{\circledcirc}\partial_{p_j}\left(\sigma^i{}_k \sigma_j{}^k\right)$$

$\lambda_{\text{◐}} = -1$
$\lambda_{\text{●}} = \pm 0$
$\lambda_{\text{◑}} = +1$