

Ising Model and Renormalization Group

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Abstract

This paper sums up the talk I gave in the Statistical Physics seminar by Prof. A. Mielke.

The aim of this paper is to give an introduction to the Renormalization Group. First, the basic concepts will be made clear on the example of a classical Ising model with temperature induced fluctuations. With the help of a mean field approximation, we will encounter a phase transition of second order. For studying this in more detail, the Exact Functional RG will be introduced, derived and solved for the simplest approximation possible, the local potential approximation. Instead of taking the continuum limit, we will start off from the path integral and just mention the analogy to the classical system.

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1 Motivation

Renormalization Groups come up in various contexts, especially in Quantum Field Theory. There are many different RGs and which one to choose depends on the problem under consideration, but they all share the same basic concept of investigating different scales of the system. In all continuous RGs there is a flow which encodes the change of physical quantities with the scale. By scale we mean an energy regime which corresponds to an inverse length. In this paper we will always set up a classical description for some high energy cut-off scale and see how the physics change by going to a macroscopic length scale and thus taking corrections due to fluctuation into account.

2 The Ising Model

Macroscopic physics is encoded in expectation values. In a statistical mechanics system, expectation values are calculated from the partition sum in the following way

$$Z = \text{Tr}[\exp(-\beta H)] \quad (1)$$

$$\langle A \rangle = \frac{1}{Z} \text{Tr}[A \exp(-\beta H)] \quad (2)$$

for some observable A . β denotes the inverse temperature and H the Hamiltonian. In the classical case, the Hamiltonian is diagonal and the trace is just the sum over the energy states of the system. The exponential is just a statistical weight factor and the trace can be generalized to some measure of the weight factor. This we will see in the functional description as the path integral.

Although the Hamiltonian is a microscopic description, the moments of the partition sum give macroscopic quantities. So, the partition sum encodes the information about fluctuations on all scales. The idea behind RG can be thought of shifting this information from the partition sum to an effective energy function.

Let us imagine a lattice and put a two-state system on every lattice site. If we think of spins, these two-state systems would take the values minus one and one. If we think of water molecules being there or not, they would take the values zero and one. For the energy function we demand, that there is only pairwise interaction with nearest neighbours. Then, alignment with neighbours and an external field both lowers the energy, opposite alignment has a positive contribution. This restricts the Hamiltonian to the following form

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} s_i s_j - h \sum_i s_i, \quad (3)$$

where the j in the first sum runs over all nearest neighbours. Note that for vanishing external field, the energy function stays unchanged when changing all two-state systems to its negative value. We say that the Hamiltonian has a $O(1)$ or mirror symmetry.

2.1 Mean field approximation

We are interested in the macroscopic behaviour of the classical Ising spin model. The expectation value of a spin is called magnetisation. To get a first intuition for this quantity, we make the approximation of a mean field. This postulates that in average each spin perfectly aligns with the mean spin m of its neighbours. To calculate the magnetisation, we choose an arbitrary coordinate system to single out one two-state system at the origin. The coupling strength is set to a coupling constant in this approximation for reasons of invariance under translation and rotation of 90° , so that the choice of the origin $i = 0$ is really arbitrary. We can now divide the Hamiltonian into one part that takes care of this one spin and its interaction with its neighbours and the other part for all other spins.

$$\begin{aligned}
H(s) &= -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} s_i s_j - h \sum_i s_i \\
&= s_0 \left(-J \sum_j s_j - h \right) - J \sum_{\langle i,j \rangle, i \neq 0} s_i s_j - h \sum_{i \neq 0} s_i \\
&\equiv s_0 \varphi + H(s')
\end{aligned} \tag{4}$$

This allows us to split and evaluate the trace in the expectation value

$$\begin{aligned}
\langle s_0 \rangle &= \frac{1}{Z} \sum_{s'} e^{-\beta H(s')} \sum_{s_0 = \pm 1} s_0 e^{-\beta s_0 \varphi} \\
&= \frac{1}{Z} \sum_{s'} e^{-\beta H(s')} * 2 \sinh(-\beta \varphi) \\
&= \frac{1}{Z} \sum_{s'} e^{-\beta H(s')} * 2 \tanh(-\beta \varphi) \cosh(-\beta \varphi) \\
&= \frac{1}{Z} \sum_{s'} e^{-\beta H(s')} * \tanh(-\beta \varphi) \sum_{s_0} e^{-\beta s_0 \varphi} \\
&= \frac{1}{Z} \sum_s e^{-\beta H(s)} \tanh(-\beta \varphi) \\
&= \langle \tanh(-\beta \varphi) \rangle \\
&= \tanh(-\beta \varphi).
\end{aligned} \tag{5}$$

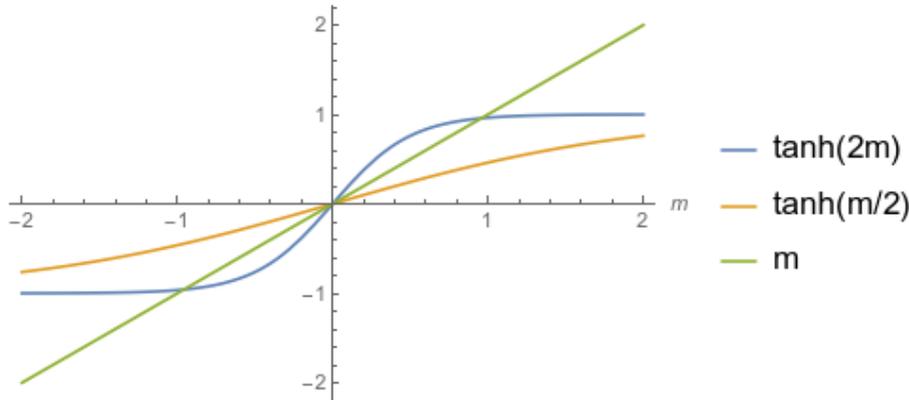


Figure 1: Solutions of equation 6 (own depiction)

We now let the mean field approximation enter. Knowing that every spin has $2d$ neighbours in arbitrary dimension d , we write

$$\langle s_0 \rangle = m = \tanh(\beta(2dmJ + h)). \quad (6)$$

Let us investigate the magnetisation for vanishing external field h . The number of solutions for this closed equation for m depends on the inverse temperature. Above a critical temperature $k_B T_c = 2dJ$ there is only one solution $m = 0$ as shown in 1. For lower temperature two solutions slide away on the line with unity slope to positive and negative magnetisations respectively. The system then chooses one of the solutions. The third solution $m = 0$ is unstable in this case, which corresponds to a maximum in the free energy. Having a non-zero solution for the magnetisation is a spontaneous symmetry breaking of the mirror symmetry in the Hamiltonian. This phase transition is continuous and we expect the magnetisation to increase as the temperature decreases as shown in figure 2. Thus, we make a power-law ansatz for the magnetisation close to the critical temperature

$$\begin{aligned} \langle s_i \rangle &= \frac{1}{Z} \text{Tr} [s_i \exp(-\beta H)] \\ &\propto |T - T_c|^\nu. \end{aligned} \quad (7)$$

Let us now see, how the information about the macrophysics can be shifted to an effective energy function and what we can learn from that. If this effective theory predicts the correct critical exponent ν , it really is an equivalent description of the macroscopic physics.

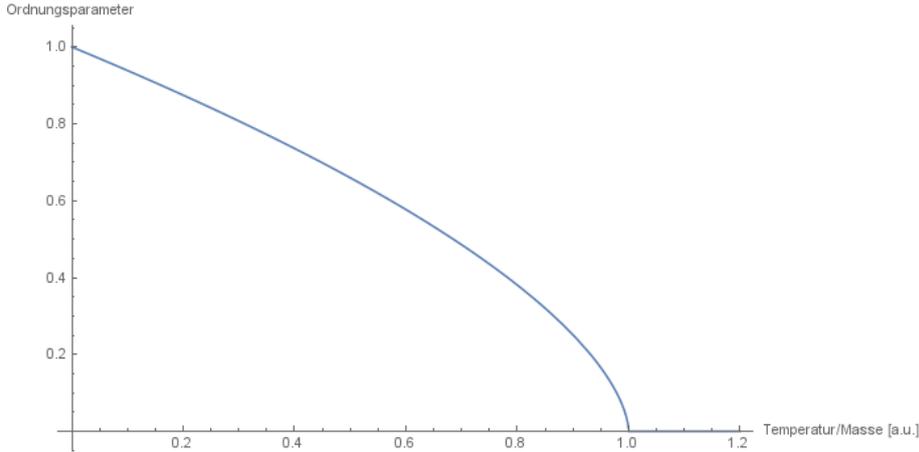


Figure 2: 2^{nd} order phase transition (own depiction)

3 Renormalization Group

3.1 Block spin transformation

For any d -dimensional grid of spins we define a block as 3^d lattice sites and neglect boundary effects. Let us now sum up the spins of a block and assign a new spin s' to the block, e.g. the sign of the sum. The coarse grained system can be described by an effective Hamiltonian $H'(s')$. Because the new Hamiltonian describes the same dynamics of the system the expectation values do not change!

$$\text{Tr} \left(s' e^{-\beta H'(s')} \right) = \text{Tr} \left(s e^{-\beta H(s)} \right) \quad (8)$$

This means that there is a certain way of manipulating the microscopic description that leaves the macroscopic description invariant! This yields the question what the transformations of the Hamiltonian look like and what we can learn about interactions and fluctuations from the change of the Hamiltonian. In the language of path integral description used in a quantum field theory, these properties can be seen very nicely. Thus, we will work through some functional definitions and calculations in the next section.

3.2 Exact Functional RG

In strict analogy to the partition sum from statistical mechanics, there is a generating functional in thermal or vacuum quantum field theory from which all expectation values can be calculated. It is defined as

$$Z[J(x)] = \int \mathcal{D}\varphi \exp \left(-S[\varphi] + \int d^d x \varphi(x) J(x) \right). \quad (9)$$

If you are not familiar with the path integral $\int \mathcal{D}\varphi$ just remember the trace in equation 1 and generalize it to some appropriate measure for the microscopic description in the exponential. The source term $J(x)$ is the analog of the external field in the classical description of the Ising model. The free energy is just the logarithm of the generating functional

$$W[J] = \ln Z[J]. \quad (10)$$

Expectation values of the field $\varphi(x)$ can be obtained by acting with a functional derivative on the generating functional and setting the source to zero

$$\phi(x) = \langle \varphi(x) \rangle = \left. \frac{\delta W[J]}{\delta J(x)} \right|_{J=0}. \quad (11)$$

The Legendre transformation to the function $\phi(x)$ as a variable is called effective action

$$\Gamma[\phi] = \sup_J \left[\int d^d x (J(x)\phi(x)) - W[J] \right]. \quad (12)$$

Note that this is still a microscopic description. The quantity of interest is the scale dependent effective action Γ_k and its change with the scale $k\partial_k\Gamma_k$. To achieve a scale dependence in the effective action we make a slight change of the generating functional

$$Z_k[J] = \int [d\varphi] \exp \left(-S[\varphi] + \int d^d x \varphi(x) J(x) - \Delta S_k[\varphi] \right) \quad (13)$$

by adding a term which is quadratic in the fields. This term can be interpreted as an effective mass.

$$\Delta S_k[\varphi] = \int \frac{d^d p}{(2\pi)^d} \varphi(p) R_k(p^2) \varphi(-p) \quad (14)$$

In the classical description equation 3 the coupling constant J is quadratic in the spin. There, we kept the coupling constant and changed the temperature so that we encountered a critical temperature $k_B T_c = 2dJ$. Equivalently, one could keep the temperature fixed and vary the term quadratic in the spins. Thus, going to smaller masses is the analog of going to smaller temperatures and we expect to encounter a critical mass.

The scale dependent effective action can now be defined as

$$\Gamma_k[\phi] = \sup_J \left[\int d^d x (J(x)\phi(x)) - W_k[J] \right] - \Delta S_k[\phi]. \quad (15)$$

To keep notation simple, we introduce a RG-time $t = \ln \frac{k}{\Lambda}$, where Λ is the UV-cutoff, the inverse lattice spacing of the microscopic grid. For the scaling evolution of the scale dependent effective action, we first compute the evolution of the scale

dependent free energy

$$\partial_t W_k[J] = -\frac{1}{Z_k[J]} \int \mathcal{D}\varphi \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \phi(p) (\partial_t R_k) \phi(-p) e^{-S[\phi] - \Delta S_k[\phi] + \int d^d x' J(x') \phi(x')} \quad (16)$$

$$= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \dot{R}_k(p^2) \langle \phi(p) \phi(-p) \rangle \quad (17)$$

$$= -\frac{1}{2} \text{Tr} \left(\dot{R}_k (G_k + \phi^2) \right) \quad (18)$$

and plug it into the following.

$$\partial_t \Gamma_k[\phi] = \int d^d x \partial_t J(x) \phi(x) - \partial_t W_k[J] - \int d^d x \frac{\delta W_k}{\delta J(x)} \partial_t J(x) - \partial_t \Delta S_k[\phi] \quad (19)$$

$$= \frac{1}{2} \text{Tr} \left(\dot{R}_k (G_k + \phi^2) \right) - \partial_t \Delta S_k[\phi] \quad (20)$$

$$= \frac{1}{2} \text{Tr} \left(\dot{R}_k G_k \right). \quad (21)$$

From

$$\frac{\delta(\Gamma_k^{(2)} + \Delta S_k)}{\delta \phi(x)} = J(x), \quad \frac{\delta W_k}{\delta J(x')} = \phi(x') \quad (22)$$

and

$$\int d^d x' \frac{\delta J(x)}{\delta \phi(x')} \frac{\delta \phi(x')}{\delta J(y)} = \delta(x - y) \quad (23)$$

it follows that the propagator G_k is the inverse operator of $\Gamma_k^{(2)} + R_k$

$$\delta^{(d)}(x - y) = \int d^d x' \frac{\delta^2 W_k[J]}{\delta J(x) \delta J(x')} \frac{\delta^2(\Gamma_k^{(2)} + \Delta S_k)}{\delta \phi(y) \delta \phi(x')} \quad (24)$$

$$= \int d^d x' G_k(x, x') (\Gamma_k^{(2)} + R_k)(x', y) \quad (25)$$

so that we can write a closed expression for Γ_k

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k \quad (26)$$

which is also known as Wetterich equation referring to [1]. The regulator is shown in figure 3. Its derivative acts like a filter so that only the propagator around the scale under consideration enters into the change of the scale dependent effective action.

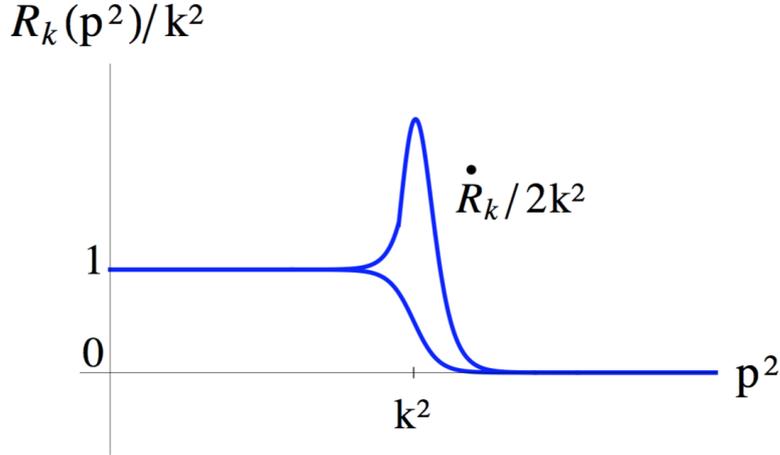


Figure 3: Regulator function and its derivative with respect to the RG scale k [2]

3.3 Fixed points

In most cases, the functional integro-differential equation (26) cannot be solved analytically. Therefore, a suitable approximation is needed. The simplest one that respects the symmetry of the Ising model is the following ansatz

$$\Gamma_k[\phi] = \int d^d x [(\partial_\mu \phi)^2 + V_{\text{eff},k}(\rho)] \quad (27)$$

$$V_{\text{eff},k}(\rho) = \sum_{i=1}^n \frac{\lambda_i(k)}{i!} \left(\frac{\phi \cdot \phi}{2}\right)^i \quad (28)$$

$$\rho = \frac{\phi \cdot \phi}{2} \quad (29)$$

which is called local potential approximation (LPA). With the regulator being proportional to a heaviside-step-function, the momentum integral just gives a volume factor which only depends on the dimension of the system. This leaves us with a functional differential equation. To make it solvable on a computer, we rescale all

quantities by k to the power of their mass dimension.

$$t = \ln \frac{k}{\Lambda} \quad (30)$$

$$\hat{\rho} = \frac{\rho}{k^{d-2}} \quad (31)$$

$$u_k(\hat{\rho}) = \frac{V_k(\rho)}{k^d} \quad (32)$$

$$\Rightarrow u'_k(\hat{\rho}) = \frac{V'_k(\rho)}{k^2}, \quad (33)$$

$$\hat{\lambda}_i(t) = \frac{\lambda_i(k)}{k^{2i-(i-1)d}}. \quad (34)$$

$$(35)$$

Plugging this into equation 26 and taking derivatives with respect to ρ of the resulting equation gives a tower of coupled differential equations for the coefficients of the effective potential. If we would cut this off by setting $\lambda_i = 0, \forall i > 2$, the evolution equations for the effective description of our system with the scale k read

$$\partial_t \lambda_1(t) = -2\lambda_1(t) - \frac{\lambda_2(t)}{2\pi^2 (1 + \lambda_1(t))^2} \quad (36)$$

$$\partial_t \lambda_2(t) = -\lambda_2(t) + \frac{3\lambda_2(t)^2}{\pi^2 (1 + \lambda_1(t))^3}. \quad (37)$$

The dimension was specified to be three in this step. The RG-time derivative of the coefficients are called beta functions and can be visualised by a stream plot (see figure 4). The Wilson-Fisher fixed point corresponds to the critical point. We see that for λ_1 close to its critical value λ_2 streams towards its critical value. But for λ_2 close to its critical value λ_1 gets carried away from its critical value. So, we encounter one attractive and one repulsive fixpoint. Away from criticality, it does not matter for the attractive component from which value the integration starts, the results will always be similar. This is the deeper reason why there can be various microscopic descriptions generating the same expectation values. If we start the numerical integration close to criticality and evaluate the minimum of the effective potential for variation of $\lambda_1(t=0)$, we will encounter a phase transition of second order from which one can read off the critical mass and the critical exponent ν . As shown for example in [3] the theoretical prediction of this procedure gives the correct numbers with accuracy better than ten percent.

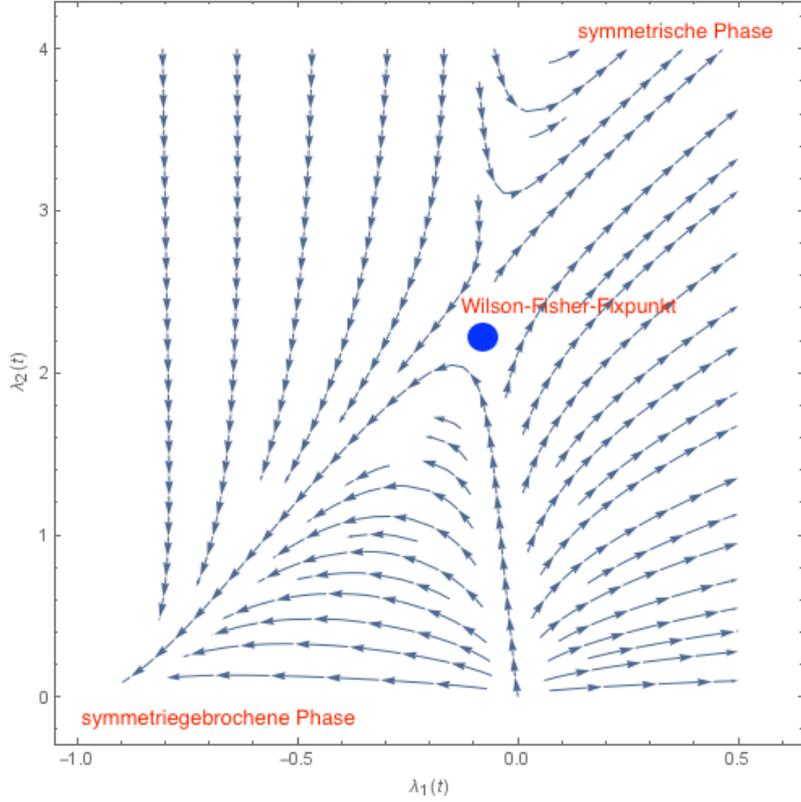


Figure 4: Evolution of the solution, arrows point towards IR (own depiction)

4 Summary

In a system that has a microscopic description, fluctuations wash out some of the information when evolving to a macroscopic description. This universal behaviour leads to universality classes that are completely characterized by the symmetries of the microscopic description, i.e. the Hamiltonian or effective action, and the spatial dimension. When shifting the information about fluctuations from the statistics, i.e. the partition sum, to an effective scale dependent description, its parameters flow through the parameter space. The Renormalization Group flow investigates the scaling behaviour of the scale dependent effective action. If there is a repulsive fixed point, the system will undergo a phase transition of second order that is characterized by its universality class. At the critical point fluctuations on all scales influence the system, whereas far away from the critical point micro- and macrophysics decouple.

References

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