Ising Model and Renormalization Group

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Motivation

- RG comes up in various contexts, eg QFT.
- Understand the concept of RG!
- Predict large scale behaviour of the Ising Model!

Overview

Statistical Mechanics

$$Z = \mathrm{Tr}[\exp(-\beta H)] \tag{1}$$

$$< A >= \frac{1}{Z}\mathrm{Tr}[A \exp(-\beta H)] \tag{2}$$

The Ising Hamiltonian

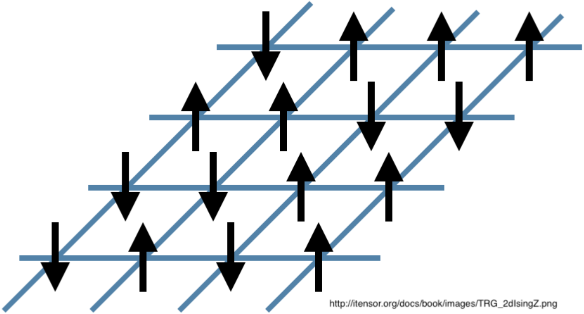

Abbildung: Lattice spins

The Ising Hamiltonian

$$H = -\frac{1}{2}\sum_{<i,j>} J_{ij} s_i s_j - h \sum_i s_i \tag{3}$$

$$< s_i > = \frac{1}{Z}\mathrm{Tr}[s_i \exp(-\beta H)] \tag{4}$$

$$= ? \tag{5}$$

Mean field

$$
\begin{aligned}
H(s) &= -\frac{1}{2}\sum_{<i,j>} J_{ij} s_i s_j - h\sum_i s_i \\
&= s_0\left(-J\sum_j s_j - h\right) - J\sum_{<i,j>,i\neq 0} s_i s_j - h\sum_{i\neq 0} s_i \\
&\equiv s_0\varphi + H(s')
\end{aligned} \tag{6}
$$

$$
\Rightarrow \langle s_0\rangle = m = \tanh\left(\beta(2dmJ + h)\right) \tag{7}
$$

Phase transitions

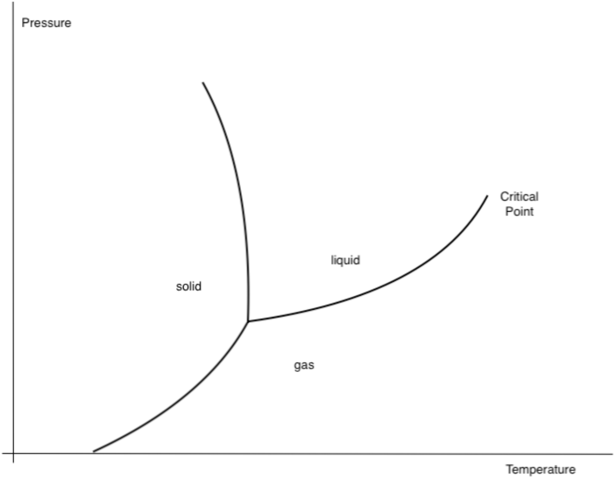

Abbildung: Phase diagram (own depiction)

Phase transitions

- 1st order
- 2nd order
- critical point
- correlation length ξ
- universality class

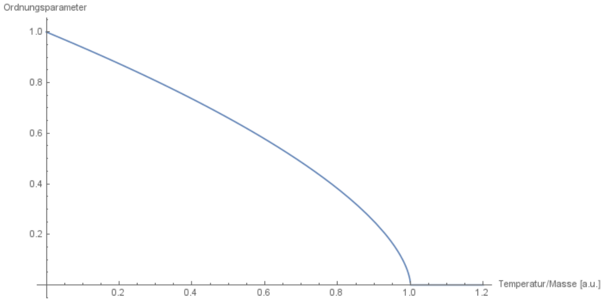

Abbildung: 2^{nd} order phase transition (own depiction)

$$\begin{aligned} \langle s_i \rangle &= \frac{1}{Z} \mathrm{Tr} \left[s_i \exp(-\beta H) \right] \\ &\propto |T - T_c|^{\nu} \end{aligned} \tag{8}$$

If the theory predicts ν correctly, it describes macroscopic physics!

Block spin

- Define a block as 3^D spins s_i.
- Sum up
- Assign spin s' to the block.

Then

$$\mathrm{Tr}\left(s' \mathrm{e}^{-\beta H'(s')}\right) = \mathrm{Tr}\left(s \mathrm{e}^{-\beta H(s)}\right) \tag{9}$$

and large scale physics remains unchanged!

QFT description

$$Z[J(x)] = \int \mathcal{D}\varphi \exp\left(-S[\varphi] + \int d^d x \varphi(x) J(x)\right) \tag{10}$$

$$W[J] = \ln Z[J] \tag{11}$$

$$\phi(x) = \langle \varphi(x) \rangle = \left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} \tag{12}$$

$$\Gamma[\phi] = \sup_J \left[\int d^d x \left(J(x)\phi(x)\right) - W[J]\right] \tag{13}$$

FRG

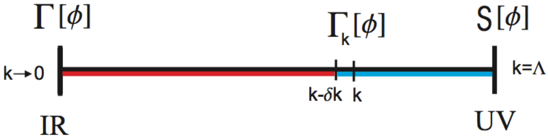

Abbildung: Sketch of RG [1]

$$Z[J] = \int [d\varphi] \exp\left(-S[\varphi] + \int d^d x \varphi(x) J(x) - \Delta S_k[\varphi]\right) \tag{14}$$

$$\Delta S_k[\varphi] = \int \frac{d^d p}{(2\pi)^d} \varphi(p) R_k(p^2) \varphi(-p) \tag{15}$$

Wetterich equation

Continuous implementation.

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\Gamma^{(2)} + R_k}\partial_t R_k\right) \tag{16}$$

The IR regulator

$$R_k(p^2) = (k^2 - p^2)\theta(k^2 - p^2) \tag{17}$$

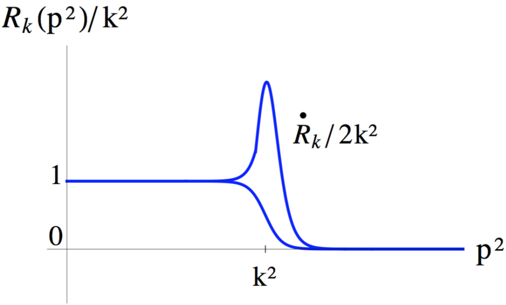

Abbildung: Regulator function and its derivative with respect to the RG scale k [1]

$$\lim_{p^2/k^2 \to 0} R_k(p^2) > 0 \tag{18}$$

$$\lim_{k^2/p^2 \to 0} R_k(p^2) \to 0 \tag{19}$$

$$\lim_{k \to \Lambda} R_k(p^2) \to \infty \tag{20}$$

Local Potential Approximation

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left[(\partial_\mu \phi)^2 + V_{\mathrm{eff},k}(\rho) \right] \tag{21}$$

$$V_{\mathrm{eff},k}(\rho) = \sum_{i=1}^{n} \frac{\lambda_i(k)}{i!} (\phi \cdot \phi)^i \tag{22}$$

$$\lambda_1'[t] = -2\lambda_1[t] - \frac{\lambda_2[t]}{2\pi^2\left(1+\lambda_1[t]\right)^2} \tag{23}$$

$$\lambda_2'[t] = -\lambda_2[t] + \frac{3\lambda_2[t]^2}{\pi^2\left(1+\lambda_1[t]\right)^3} \tag{24}$$

Fixed points

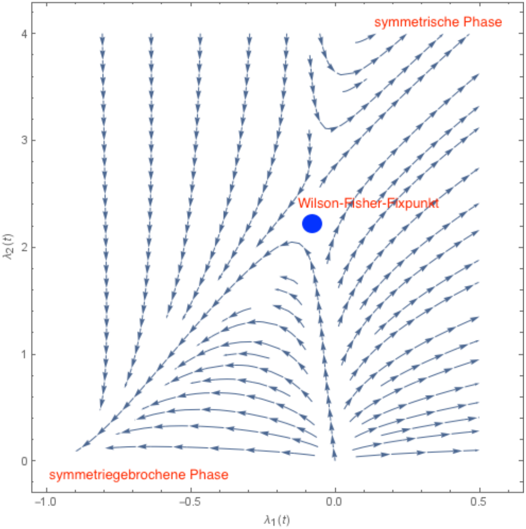

Abbildung: Evolution of the solution, arrows point towards IR (own depiction)

Take home message

- The Ising model defines a universality class for a $O(1)$-symmetric Hamlitoniam.
- RG flows investigate scaling behaviour.
- At critical point: fluctuations on all scales
  Away from criticality: decoupling of macro- from microphysics!

Outlook

- Generalization in the $O(N)$-model: Ising, XY, Heisenberg, Higgs, polymers
- Theories for spiking neural networks and deep learning are Ising-like

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