The Mermin-Wagner theorem and the Kosterlitz-Thouless transition

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SSB and phase transition

- Symmetry $\leftrightarrow$ Hamiltonian invariant under transformation
- Physical states not necessarily invariant, e.g. Ising model
- Leads to phase transition, e.g.

$$\langle m \rangle = 0 \xrightarrow{T<T_c} \langle m \rangle \neq 0$$

*However, **no** equivalence between the two.*

Theorem (Mermin-Wagner)

There is no spontaneous symmetry breaking in systems with short-range interactions and a continuous symmetry in $d \leq 2$.

$$\mathcal{H} = -\sum_{i,j} J_{ij}\vec{s}_i \cdot \vec{s}_j - h\sum_i (\vec{s}_i)_z e^{-i\vec{k}\cdot\vec{r}_i}$$

$$\frac{1}{2}\langle\{\hat{A},\hat{A}^\dagger\}\rangle \geq k_B T|\langle[\hat{C},\hat{A}]\rangle|^2/\langle[[\hat{C},\hat{H}],\hat{C}^\dagger]\rangle$$

$$|s_z| \leq \frac{\text{const.}}{T^{1/2}} \frac{1}{|\ln(|h|)|^{1/2}}$$

The XY model

Like the Ising model in 2D

$$\begin{aligned}\mathcal{H} &= -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j, \quad \vec{s}_i = (\cos\theta_i, \sin\theta_i) \\ &= -J \sum_{\langle ij \rangle} \cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j \\ &= -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)\end{aligned}$$

Use the Schwarz inequality

$$\langle AA^* \rangle \geq \frac{|\langle AB^* \rangle|^2}{\langle BB^* \rangle}$$

$$A = \frac{1}{N} \sum_j e^{-i\vec{q}\cdot\vec{r}_j} \sin \phi_j$$

$$B = \frac{1}{N} \sum_l e^{-i\vec{q}\cdot\vec{r}_l} \frac{\partial H}{\partial \phi_l}$$

Correlation length (high temperature)

$$\begin{aligned}
\langle \vec{s}_0 \cdot \vec{s}_r \rangle &= \langle \cos\left(\theta_0 - \theta_r\right) \rangle \\
&= \frac{1}{Z} \prod_{i=1}^{N} \left(\int_0^{2\pi} \frac{d\theta_i}{2\pi} \right) \cos(\theta_0 - \theta_r) e^{K \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)} \\
&= \frac{1}{Z} \prod_{i=1}^{N} \left(\int_0^{2\pi} \frac{d\theta_i}{2\pi} \right) \cos(\theta_0 - \theta_r) \prod_{\langle i,j \rangle} \left[1 - K \cos(\theta_i - \theta_j) + \mathcal{O}(K^2) \right]
\end{aligned}$$

$$\int_0^{2\pi} \frac{d\theta_1}{2\pi} \cos(\theta_1 - \theta_2) = 0$$

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_3) = \frac{1}{2} \cos(\theta_1 - \theta_3)$$

Correlation length (low temperature)

Use spin wave approximation

$$
\begin{aligned}
-\beta\mathcal{H} &= \beta J \sum_{\langle ij \rangle} \left(1 - \frac{1}{2}(\theta_i - \theta_j)^2\right) \\
&= -\frac{K}{2} \int (\nabla\theta)^2
\end{aligned}
$$

$$
\begin{aligned}
\langle \vec{s}_0 \cdot \vec{s}_r \rangle &= \langle \cos(\theta_0 - \theta_r) \rangle \\
&= \langle e^{i(\theta_0 - \theta_r)} \rangle
\end{aligned}
$$

Gaussian integration

$$
\langle e^{i(\theta_r - \theta_0)} \rangle = e^{-\frac{1}{2}\langle(\theta_r - \theta_0)^2\rangle}
$$

Phase transition

- High T: $\langle \vec{s}_0 \cdot \vec{s}_r \rangle \sim \exp(-r/\xi), \quad \xi \approx \frac{1}{\ln(2/K)}$
- Low T: $\langle \vec{s}_0 \cdot \vec{s}_r \rangle \sim r^{-1/2\pi K}$

Spin wave approximation has to break down at some T. Kosterlitz and Thouless proposed topological defects as the origin of this phase transition.

Vortices

$$\theta(\vec{r}) = n\phi + c, \quad \text{winding number } n$$

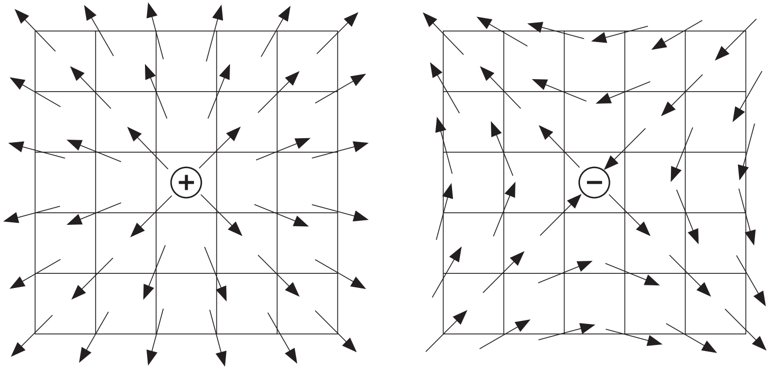

Image: *Statistical Physics of Fields (Kardar)*

Vortex energy and entropy

$$(\nabla\theta)_r = \frac{\partial\theta}{\partial r} = 0, \quad (\nabla\theta)_\phi = \frac{1}{r}\frac{\partial\theta}{\partial\phi} = \frac{n}{r}$$

$$E = \frac{J}{2}\int \frac{n^2}{r^2} r dr\, d\phi = n^2 \pi J \ln\frac{R}{r_0} + E_C$$

$$S = \ln\left(\frac{R}{r_0}\right)^2 = 2\ln\left(\frac{R}{r_0}\right)$$

(Un)binding of vortices

Free energy change from one vortex

$$\Delta F_1 = (\pi J - 2T) \ln \frac{R}{r_0}$$
$$T_{KT} = \pi J/2$$

Simple approximation, neglects vortex interaction. More sophisticated RG methods exist.

$$E = \left(\sum_i n_i \right)^2 \pi J \ln \frac{R}{r_0} + E_C$$

Mapping to 2D Coulomb gas

Setting $\vec{v} \equiv \nabla\theta$

$$\oint \vec{v} \cdot d\vec{r} = \int_0^{2\pi} \frac{1}{r} r d\phi = 2\pi$$
$$= \int (d^2x\,\hat{z}) \nabla \times \vec{v}$$

It follows that

$$\nabla \times \vec{v} = 2\pi \sum_i n_i \delta(\vec{r} - \vec{r}_i) \equiv 2\pi N(\vec{r})$$

Mapping to 2D Coulomb gas (cont.)

Expressing $\vec{v}$ in cartesian coordinates

$$v_x = -\frac{y}{r^2}, \quad v_y = \frac{x}{r^2}$$

Introduce scalar field $\psi = -\log(r/r_0)$

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}$$

Or generally

$$\psi(\vec{r}) = -\sum_i \log \frac{|\vec{r} - \vec{r}_i|}{\vec{r}_0}$$

Mapping to 2D Coulomb gas (cont.)

The energy then is

$$\begin{aligned} E &= \frac{J}{2}\int (v_x^2 + v_y^2)dx\,dy = \frac{J}{2}\int (\nabla\psi)^2 dx\,dy \\ &= -\frac{J}{2}\int (\psi\nabla^2\psi)dx\,dy \\ &= -\pi J\sum_{i\neq j} n_i n_j \log\frac{|\vec{r}_i - \vec{r}_j|}{\vec{r}_0} + E_C \end{aligned}$$

References

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