Recall that $f(k,\theta) = \frac{1}{k} \sum_{l \geq 0} (2l+1)\, a_l(k)\, P_l(\cos\theta)$

$a_l(k) = e^{i\delta_l} \sin\delta_l .$

Assume $v(r) = 0$ for $r > a$. Then for $r > a$, ⊛

$$R_l = R_l^{>} = e^{i\delta_l}\left(j_l(kr)\cos\delta_l - n_l(kr)\sin\delta_l \right)$$ $\frac{1}{2}$?

Because we had to fix one coeff. to cancel the e^{-ikr} part of the plane wave, we compose the continuity condition as

$$\frac{d}{dr} \log R_l^{>} \Big|_{r=a} = \alpha_l$$

$$\alpha_l = \frac{d}{dr} \log R_l^{<} \Big|_{r=a}.$$

⊛ asymptotically, $\sin\left(kr - l\frac{\pi}{2} - \delta_l(k)\right)$
$\delta_l = 0$ for $U = 0$

← shift, as compared to the free case.

(2) This implies

$$\tan \delta_\ell = \left. \frac{\frac{d}{dr} j_\ell(kr) - \alpha_\ell j_\ell(kr)}{\frac{d}{dr} n_\ell(kr) - \alpha_\ell n_\ell(kr)} \right|_{r=a}$$

Let $\rho_0 = k \cdot a$. Assume $\rho_0 \ll 1$.
Then

$$j_\ell(\rho_0) = \frac{1}{(2\ell+1)!!} \rho_0^{\ell} + \dots \qquad \ell=0: \ \frac{\sin\rho}{\rho}$$

$$n_\ell(\rho_0) = -\frac{(2\ell-1)!!}{\rho^{\ell+1}} + \dots \qquad -\frac{\cos\rho}{\rho}$$

and

$$\tan \delta_\ell = -\rho_0^{2\ell+1} \underbrace{\frac{1}{(2\ell+1)!!(2\ell-1)!!}}_{\frac{2\ell+1}{((2\ell+1)!!)^2}} \; \frac{\frac{k}{\rho_0}\ell - \alpha_\ell}{-k\frac{(\ell+1)}{\rho_0} - \alpha_\ell} .$$

$$\frac{\ell - a\alpha_\ell}{\ell+1+a\alpha_\ell}$$

Thus for $k \to 0$, $a_\ell(k) \sim k^{2\ell+1}$,

$$a_\ell(k) = -k^{2\ell+1} \lambda_\ell + O(k^{2\ell+3})$$

ℓ-scattering "length"

$\ell=0$: s-wave: $a_0(k) = -k\lambda_0 + O(k^3)$

$$f_0(k,\Theta) = -\lambda_0$$

s-wave scattering length.

Since the expansion in ρ_0 converges rapidly, it suffices to take into account l up to ρ_0^2

[if $l \gg \rho_0^2/2$ then $\frac{\rho_0^2}{2l} \ll 1$, hence $\frac{c_{l+1}\,\rho_0^{2l+3}}{c_l\,\rho_0^{2l+1}} = \rho_0^2 \cdot \underbrace{\frac{c_{l+1}}{c_l}}_{\sim \frac{1}{2l}} \ll 1$]

Thus for $\rho_0 \ll 1$, $l = 0$ suffices.

$$\tan\delta_0 = -\rho_0 \cdot \frac{-\alpha_0}{-\alpha_0 - \frac{k}{\rho_0}} = -\rho_0^2 \, \frac{\alpha_0 \rho_0}{k + \alpha_0 \rho_0}$$

Ex 1 Case of hard ~~spheres~~ balls. $U(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$

This means that $u_0(r) = 0$ at $r = a$.

general: $R_l(a) = 0$, $\alpha_l = \infty$.

$$\Rightarrow \tan\delta_l = \frac{j_l(ka)}{n_l(ka)}$$

$$\Rightarrow \delta_0 = -ka, \quad \text{so} \quad \boxed{\lambda_0 = a}$$

↑ negative for repulsive potential!

Thus: at small enough k, every repulsive potential looks like a hard ball, with radius = s-wave scattering length.

"if you have little energy, every game becomes handball"

Effective range:

$$k \cdot \cot\delta_0(k) = -\frac{1}{\lambda_0} + \frac{1}{2} r_0 k^2 \ldots$$

effective range

(4) Then only $l=0$ contributes, so

$$f(k,\theta) = \frac{1}{k}(2\cdot 0+1)\cdot e^{i\delta_0}\sin\delta_0 \qquad \delta_0 = -ak$$

$$\frac{d\sigma}{d\Omega}\left(=\frac{1}{k^2}\sin^2\delta_0\right) = a^2$$

$$\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = 4\pi a^2$$

classical cross-section: πa^2.

Note that for $l=0$, the u-eq. is

$$-u_0'' + U\cdot u_0 = k^2 u_0$$

so for $U=0$ and $k=0$, this reduces to

$$u_0'' = 0, \quad \text{thus} \quad u_0 = B + Ar,$$

and $R_0 = \frac{u_0}{r} = A + \frac{B}{r}$

We may choose $A=1$. The condition that $R_0(a)=0$ then leads to $B=-a$,

$$R_0(r) = 1 - \frac{a}{r}.$$

Ex: 2: Spherical well

$$U(r) = -U_0 \, \mathbb{1}_{r \leq a} \qquad (U_0 > 0)$$

Then
$$-u_\ell'' + \left(\frac{\ell(\ell+1)}{r^2} - U_0 \, \mathbb{1}_{r\leq a}\right) u_\ell = k^2 u_\ell$$

so
$$-u_{\ell,\gtrless}'' + \left(\frac{\ell(\ell+1)}{r^2} - k_{\gtrless}^2(r)\right) u_\ell = 0$$

$$k_>^2 = k^2 \qquad k_<^2 = k^2 + U_0 > k^2.$$

regular solution inside:

$$R_\ell^<(r) = j_\ell(k_< r), \qquad \alpha_\ell = k_< \frac{\hat{j}_\ell'(k_< a)}{\hat{j}_\ell(k_< a)}$$

If we do not restrict to very small $k_< a$, then the denominator in the eq. for $\tan \delta_\ell$ may vanish:

$$\ell + 1 + a \cdot \alpha_\ell(k) = 0 \qquad (R)$$

Then $\delta_\ell(k) = \frac{\pi}{2} + n\pi$, and $\sigma_\ell(k) = \frac{4\pi}{k^2}(2\ell+1)$ becomes maximal.

(R) is called resonance condition.

[resonance happens for general potentials with bound states]

⑥ For the spherical well, bound state solutions are pieced together from $\left(\epsilon = \frac{-2mE}{\hbar^2}\right)$ $(\epsilon > 0)$

$$R_<(r) = A \cdot j_\ell(\kappa_< r) \qquad (r<a), \quad \kappa_<^2 = U_0 + \epsilon$$

$$R_>(r) = B\, h_\ell^{(+)}(i\kappa_> r) \qquad r>a, \quad \kappa_>^2 = \epsilon$$

This is the only possibility because j_ℓ is the only sol. that is regular at zero, and $h_\ell^{(+)}(i\kappa_> r)$ is the only one that decays fast enough to have a square integrable solution.

The continuity condition is.

$$\kappa_< \frac{d}{d\rho} \log j_\ell(\rho)\Big|_{\rho=\kappa_< a} = i\kappa_> \frac{d}{d\rho} \log h_\ell^{(+)}(\rho)\Big|_{\rho = i\kappa_> a}$$

<u>$\ell = 0$:</u> $u_0(r) = \begin{cases} A \sin(\kappa_< r) & r<a \\ B\, e^{-\kappa_> r} & r>a \end{cases}$

$$\rightarrow \cot(\kappa_< a) = -\frac{\kappa_>}{\kappa_<}$$

bound states exist for $U_0 \geq \left(\frac{\pi}{2}\right)^4 \cdot \frac{1}{a^2}$

Consider the case

$$k_> a \ll l \ll k_< a$$

i.e. resonant scattering off a deep well, at small energies. Then $k_> a - l\frac{\pi}{2} - \delta_l(k) \approx -l\frac{\pi}{2}$, so it is $|\cdot| \gg 1$, hence we can use the large ρ — asymptotic formula

$$j_l(\rho) \sim \frac{1}{\rho}\sin\left(\rho - l\frac{\pi}{2}\right)$$

Then

$$\alpha_l(k) = \dots$$

so that $$\frac{l}{k_< a} = -\cot\left(k_< a - \frac{l\pi}{2}\right)$$

$$k_< a - l\frac{\pi}{2} \approx \left(n+\frac{1}{2}\right)\pi + \frac{1}{k_< a}$$

Expansion of

$$\tan\delta_l = \frac{2l+1}{((2l+1)!!)^2}(ka)^{2l+1}\frac{l - a\alpha_l}{l+1+a\alpha_l}$$

gives

$$\tan\delta_l = -C\cdot\frac{(ka)^{2l+1}}{E-E_R} + O(\dots)$$

with $$C = \frac{-1}{[(2l-1)!!]^2\, a\alpha_l'(E_R)} > 0.$$

consequence of causality

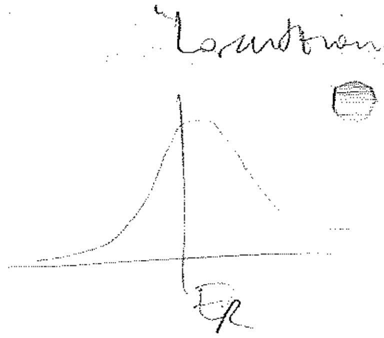
(8)

Let $\Gamma_k = 2(ka)^{2\ell+1} \cdot C$.

Then

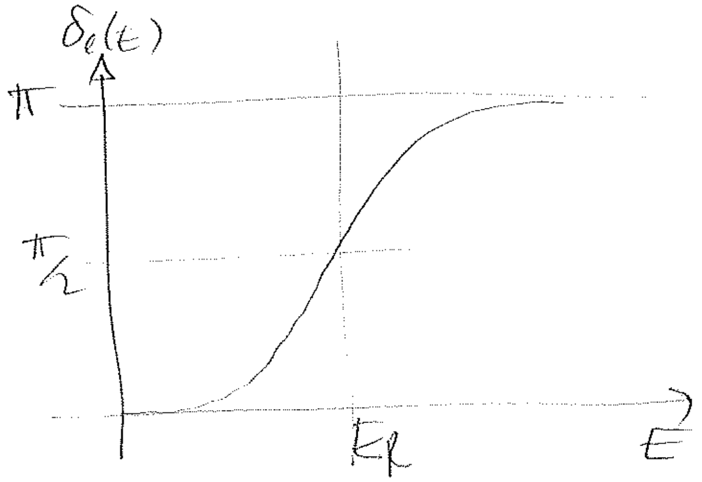
$$\sigma_\ell = |a_\ell(k\Theta)|^2 = \frac{4\pi(2\ell+1)}{k^2} \frac{(\Gamma_k/2)^2}{(E-E_R)^2 + (\Gamma_k/2)^2}$$

(Breit-Wigner-Formel)



$$a_\ell(k) = \frac{\tan\delta_\ell}{k(1 - i\tan\delta_\ell)}$$

$$= \frac{-\Gamma_k/2}{E - E_R + i\frac{\Gamma_k}{2}}$$



$a_\ell(k)$ has a pole at $E = E_R - i\frac{\Gamma_k}{2}$
(in the lower half plane)

The variable-phase equation.

There is some ambiguity in the def. of the phases. Natural idea is to say:

$k \to \infty$: kinetic energy dominant, so

require $\delta_\ell(k) \xrightarrow[k\to\infty]{} 0 \mod \pi$

Put $\delta_\ell(\infty) = 0$, and require $\delta_\ell(k)$ to be continuous in k.

Calogero (1967) derived a ~~DE~~ for a function $\delta_\ell(k,r)$ defined as phase shifts of a potential

$$U_r(r') = U(r')\, 1_{r' \leq r}$$

He showed [$\delta_\ell(k,0)$ assumed to be small or zero)

$$\frac{\partial}{\partial r}\delta_\ell(k,r) = -\frac{1}{k}\,U(r)\Big[\tilde{j}_\ell(kr)\cos\delta_\ell(k,r) - \tilde{n}_\ell(kr)\sin\delta_\ell(k,r)\Big]^2$$

[NB: if $U \geq 0$, one sees that increasing the potential decreases the phase shift]

For $\ell = 0$:

$$\delta_0(k) = -\frac{1}{k}\int_0^\infty dr\, U(r)\sin^2\big[kr + \delta_0(k,r)\big]$$

$\delta_0(k) = \delta_0(k,\infty)$

Coulomb scattering

$V(r) = \frac{Ze^2}{r}$. Define n by

$$2nk = \frac{2m}{\hbar^2} Ze^2$$

Asymptotics changes to

$$R_\ell(r) \sim \frac{1}{r} e^{\pm i(kr - n \cdot \log(kr))} + O\left(\frac{1}{r^2}\right)$$

easily verified since $rR_\ell = u_\ell$ satisfies

$$-u_\ell'' + \left(\cancel{\frac{\ell(\ell+1)}{r}} + 2nk\right) u_\ell = k^2 u_\ell$$

$O\left(\frac{1}{r}\right)$

Thus $$\psi = e^{ik \cdot x_3} + f(k,\theta) \frac{e^{ikr}}{r} e^{in\log(kr)}$$

Detailed sol. with confluent hypergeometric fct ${}_1F_1$ gives

$$f(k,\theta) = \frac{-n}{2k \sin^2\frac{\theta}{2}} e^{i(2\eta_0 - n\log(2\sin^2\frac{\theta}{2}))}$$

$$\frac{d\sigma}{d\Omega} = |f(k\theta)|^2 = \frac{n^2}{4k^2 \sin^2\frac{\theta}{2}} = \left(\frac{Ze^2}{2mv^2 \sin^2\frac{\theta}{2}}\right)^2$$

where $v = \frac{\hbar k}{m}$ corresponds to a classical velocity

This result, Rutherford's formula for the differential cross section for Coulomb scattering, agrees with the classical mechanics result.