AQT, June 3rd, 2011

Recall: Density operator ("density matrix")
= trace-class operator ρ with the properties

(1) $\rho = \rho^*$

(2) $\rho \geq 0$

(3) $\mathrm{Tr}\,\rho = 1$

ρ trace-class $\longrightarrow$ the spectrum of ρ consists of eigenvalues that can accumulate only at zero.

(1) $\rightarrow$ $\exists$ $\rho \phi_k = p_k \phi_k$, $p_k \in \mathbb{R}$, and

$$\rho = \sum_k p_k P_{\phi_k} = \sum_{k \geq 1} p_k |\phi_k\rangle\langle\phi_k|$$

(2) $\rightarrow$ $p_k \geq 0$ $\forall k$.
(may order them $p_1 \geq p_2 \geq p_3 \geq \dots$)

(3) $\sum p_k = 1$.

$\rightarrow$ the p_k are probabilities.

$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$

ψ on $\mathcal{H}$, $\|\psi\| = 1$,

$\langle \psi | A \otimes \mathbb{1}_E \psi \rangle = \mathrm{Tr}(\rho A)$

ρ a density matrix on $\mathcal{H}_S$.

②

[if $\psi = \sum \psi_{i,j} \; s_i \otimes e_j$

ONBasis S   ONB on $\mathcal{H}_\mathcal{E}$

then $\rho_{ii'} = \langle s_i | \rho \, s_{i'} \rangle = \sum_j \psi_{i,j} \overline{\psi_{i',j}}$]

Thus: define a QM state as a density matrix.

This notion is robust under restriction to subspaces by partial traces.

"$\rho_S = \mathrm{Tr}_\mathcal{E} \, \rho$"

meaning:

$$\mathrm{Tr}(\rho \, A \otimes \mathbb{1}_\mathcal{E}) = \mathrm{Tr}(\rho_S A)$$

for every A on $\mathcal{H}_S$.

The set of states is convex:
If ρ_0 and ρ_1 are DM and $\lambda \in [0, 1]$,
then
$$\rho_\lambda = (1-\lambda)\rho_0 + \lambda \rho_1$$
is a DM as well. $^{x)}$

Def. A DM ρ describes a pure state
if $\rho^2 = \rho$, otherwise a mixed state.

$\rho = P_\Psi$ (projection to a normalized $\Psi \in \mathcal{H}$)
is a pure state since for projections
$$P_\Psi^2 = P_\Psi .$$
This means that in $\rho = \sum_k p_k P_{\phi_k}$
only one of the p_k's is nonzero:
$$p_{k_0} = 1, \quad p_k = 0 \quad \forall k \neq k_0.$$
Mixed state: $p_k > 0$ for more than one k.
$\Rightarrow p_k^2 < p_k$ for all nonzero p_k, since $0 < p_k < 1$
$$\Longrightarrow \rho^2 < \rho.$$

x) extremal states ...

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Von Neumann entropy.

$$S(\rho) = - \mathrm{Tr}(\rho \ln \rho)$$

[sometimes k_B ...]

spectral rep. of ρ

$$\Rightarrow \quad S(\rho) = - \sum_k p_k \ln p_k \qquad (0 \cdot \log 0 := 0)$$

→ up to a factor, $S(\rho)$ is Shannon's information entropy of the probability measure given by $p_1, p_2, \ldots$

$S(\rho) \geq 0$ by definition.

$S(\rho) = 0 \iff \exists k_0 : p_{k_0} = 1, \; p_k = 0 \;\; k \neq k_0.$

pure state (outcome known with certainty)

Thus pure states are characterized by a vanishing vN entropy

$S(\rho) > 0 \Rightarrow \rho$ mixed state.

$S(\rho)$ maximal $\iff \forall k \in \{1 \ldots N\} : p_k = \frac{1}{N}$

if $\dim \mathcal{H} = N$

(equipartition, "maximal uncertainty" of outcome)

Entanglement

A pure state $|\psi\rangle\langle\psi|$ on a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is entangled

$\Longleftrightarrow$ ψ is not simply a tensor product $\phi_1 \otimes \phi_2$.

[similar def's for $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$].

In general, very hard to determine if ψ is entangled. For bipartite systems, there is a tool for determining this, the Schmidt decomposition:

Let $\rho = |\psi\rangle\langle\psi|$ and $\rho_1 = \mathrm{Tr}_2(\rho)$, $\rho_2 = \mathrm{Tr}_1(\rho)$.

Then

1. ρ_1 and ρ_2 have the same eigenvalues

2. If $(e_i)_i$ is an ONB of $\mathcal{H}_1$ consisting of EV of ρ_1, then there are $\phi_i \in \mathcal{H}_2$ such that

$$\psi = \sum_i \sqrt{\lambda_i}\, e_i \otimes \phi_i$$

and ψ is entangled if there is more than one term in this sum.

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Proof.

$$\psi = \sum_{i,j} \psi_{i,j} \; e_i \otimes \underbrace{f_j}_{\text{ONB of } \mathcal{H}_2}$$

$$= \sum_i e_i \otimes \underbrace{\sum_j \psi_{ij} f_j}_{=: \tilde{\varphi}_i}$$

$$\rho_1 = \mathrm{Tr}_2 |\psi\rangle\langle\psi|$$

$$\langle e_i | \rho_1 e_{i'} \rangle = \sum_j \langle e_i \otimes f_j \,|\, P_\psi \; e_{i'} \otimes f_j \rangle$$

$$= \sum_{j,k,k'} \langle e_i \otimes f_j | e_k \otimes \tilde{\varphi}_k \rangle \langle e_{k'} \otimes \tilde{\varphi}_{k'} | e_{i'} \otimes f_j \rangle$$

$$= \sum_j \langle f_j | \tilde{\varphi}_i \rangle \langle \tilde{\varphi}_{i'} | f_j \rangle$$

$$= \langle \tilde{\varphi}_{i'} | \tilde{\varphi}_i \rangle$$

since $(e_i)_i$ is an ONB of ρ_1, we have

$$\langle e_i | \rho_1 e_{i'} \rangle = \delta_{ii'} \lambda_i \quad , \quad \lambda_i \geq 0$$

$$\rightarrow \langle \tilde{\varphi}_{i'} | \tilde{\varphi}_i \rangle = \lambda_i \delta_{ii'}$$

remove those i with $\lambda_i = 0$. Then $\frac{1}{\sqrt{\lambda_i}} \tilde{\varphi}_i =: \varphi_i$ is an orthonormal set in $\mathcal{H}_2$.

and $\psi = \sum_i \sqrt{\lambda_i}\, e_i \otimes \phi_i$

$\rho_2 = Tr_1 P_\psi$.

$$\langle \phi_j | \rho_2 \phi_{j'} \rangle = \sum_k \langle e_k \otimes \phi_j | P_\psi \, e_k \otimes \phi_{j'} \rangle$$

$$= \sum_{k,i,i'} \langle e_k \otimes \phi_j | \sqrt{\lambda_i}\, e_i \otimes \phi_i \rangle \langle \sqrt{\lambda_{i'}}\, e_{i'} \otimes \phi_{i'} | e_k \otimes \phi_{j'} \rangle$$

$$= \sum_k \lambda_k \cdot \langle \phi_j | \phi_k \rangle \langle \phi_k | \phi_{j'} \rangle$$

$$= \delta_{jj'} \lambda_j$$

so the ϕ_j are an eigenbasis of ρ_2.

Spin-$\frac{1}{2}$

Then $\mathcal{H} \cong \mathbb{C}^2$.

A basis for all operators are $\mathbb{1}, \sigma_1, \sigma_2, \sigma_3$,
where the σ_i are the Pauli matrices (sometimes $\sigma_0 := \mathbb{1}$)

so any $A = a_0\sigma_0 + a_1\sigma_1 + \ldots + a_3\sigma_3 = \sum_{i=0}^{3} a_i\sigma_i$.

$A = A^* \Rightarrow a_i \in \mathbb{R} \ \forall i$.

$e^{iA} = e^{ia_0} e^{i\vec{a}\cdot\vec{\sigma}}$

$\Rightarrow$ for Hamiltonians, $a_0 = 0$. $H = \vec{h}\cdot\vec{\sigma}$.

ρ a DM. Write $\rho = \frac{1}{2}(r_0\mathbb{1} + \vec{r}\cdot\vec{\sigma})$

$\rho = \rho^* \Rightarrow r_i$ real.

Since $(\vec{r}\cdot\vec{\sigma})^2 = \vec{r}^2\cdot\mathbb{1}$, $\vec{r}\cdot\vec{\sigma}$ has EV $\pm|\vec{r}|$.
$\Rightarrow \rho$ has EV $\frac{1}{2}(1 \pm |\vec{r}|)$

$\mathrm{Tr}\,\sigma_i = 0 \Rightarrow \mathrm{Tr}\,\rho = r_0$. Thus $r_0 = 1$,

$|\vec{r}| \leq 1$ must hold

If $|\vec{r}| = 1$, $p_1 = 1$, $p_2 = 0$: pure state

If $|\vec{r}| < 1$, both $p_i \neq 0 \Rightarrow$ mixed state.

equipartition is when $|\vec{r}| = 0$, then

$\rho = \frac{1}{2}\cdot\mathbb{1}$

the state is uncorrelated.

$\frac{1}{2} \otimes \frac{1}{2}$ and the Bell states

Know from angular momentum addition that $\frac{1}{2} \otimes \frac{1}{2} = \underset{\text{singlet}}{0} \oplus \underset{\text{triplet}}{1}$

so besides the ONB $e_i \otimes e_j$ $i, j \in \{1, 2\}$

can also take an ONB corresponding to this decomposition into irreducible pieces.

The normalized basis vectors are usually called Bell states in quantum information theory.

singlet:

$$|\psi_-\rangle = \frac{1}{\sqrt{2}} \left(|\overset{+-}{\uparrow\downarrow}\rangle - |\overset{-+}{\downarrow\uparrow}\rangle \right)$$

where $|\uparrow\downarrow\rangle = |\uparrow\rangle \otimes |\downarrow\rangle$ $\quad |\overset{+}{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |\overset{-}{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

are the eigenvectors of σ_3.

triplet:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \right)$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right)$$

Exercise: show that these states are orthonormal.

These states are entangled states (maximally entangled), but the (reduced) DM for subsystem 1 is for all of them:

$$\rho = \frac{1}{2} \cdot \mathbb{1}_2$$

i.e. it describes an uncorrelated state. (maximal entropy)

e.g. for ψ_-: let $A \in M_2(\mathbb{C})$

$\langle + | - \rangle = 0$

$$\langle \psi_- | A \otimes \mathbb{1} \psi_- \rangle =$$

$$= \frac{1}{2} \Big[\langle +- | A \otimes \mathbb{1} | +- \rangle - \langle +- | A \otimes \mathbb{1} | -+ \rangle - \langle -+ | A \otimes \mathbb{1} | +- \rangle + \langle -+ | A \otimes \mathbb{1} | -+ \rangle$$

$$= \frac{1}{2} \left(A_{++} + A_{--} \right) = \frac{1}{2} \operatorname{Tr}(A)$$

$$= \operatorname{Tr}\left(\tfrac{1}{2} \mathbb{1} \cdot A\right)$$

True for all $A \implies \rho = \frac{1}{2} \mathbb{1}_2$.

The singlet state is well realizable in practice and it exhibits perfect anticorrelations:

$$(\vec{\sigma} \otimes \mathbb{1})\psi_- = -(\mathbb{1} \otimes \vec{\sigma})\psi_-$$

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. (Exercise)

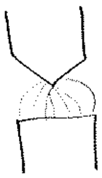
Consequence:

$$\langle \psi_- | (\vec{a}\cdot\vec{\sigma} \otimes \vec{b}\cdot\vec{\sigma})\psi_- \rangle = -\vec{a}\cdot\vec{b}$$

for all $\vec{a}, \vec{b} \in \mathbb{R}^3$.

Exercise: check these relations.

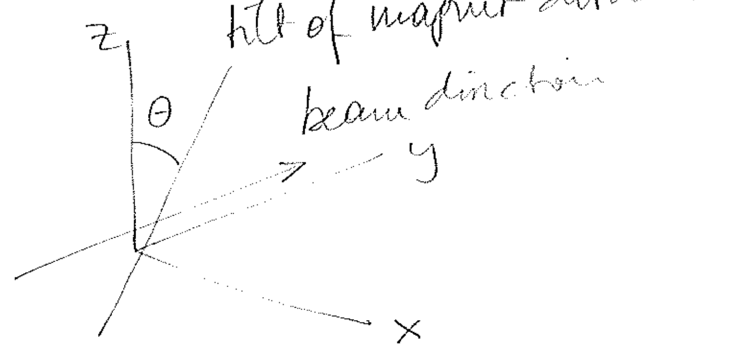
Observables of the type $\vec{a}\cdot\vec{\sigma}$ occur in (idealized) Stern-Gerlach devices.



$$H = \frac{p^2}{2m} + \mu \cdot \vec{\sigma} \cdot \vec{B}(\vec{x})$$

↑ strongly inhomogeneous, leads to splitting of beam.

Use coordinate system



$$\vec{n}(\theta) = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

Then $\vec{n}(\theta)\cdot\vec{\sigma} = \sin\theta\,\sigma_1 + \cos\theta\,\sigma_3 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

This matrix has eigenvectors

$$\vartheta_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad \vartheta_- = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

to EV $\quad +1 \qquad\qquad -1$.

[$\sigma_3 e^{i\theta\sigma_2} = \sigma_3\cos\theta + i\sin\theta\,\underbrace{\sigma_3\sigma_2}_{i\varepsilon_{321}\sigma_1} = \sigma_1\sin\theta + \sigma_3\cos\theta$

$\sigma_3 e^{i\theta\sigma_2} \underset{\substack{\uparrow \\ \{\sigma_3,\sigma_2\}=0}}{=} e^{-\frac{i}{2}\theta\sigma_2}\,\sigma_3\, e^{\frac{i}{2}\theta\sigma_2}$ $= U\sigma_3 U^{-1}$; thus $U = e^{-\frac{i}{2}\theta\sigma_2}$ diagonalizes it

| → EV are columns of U

$$U = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

Because ψ_- is a state with angular momentum $l = 0$, it is rotationally invariant. as a consequence,

$$\psi_- = \frac{1}{\sqrt{2}}\left(0_+ \otimes 0_- - 0_- \otimes 0_+ \right)$$

$$= \frac{1}{\sqrt{2}}\left(\vartheta_+ \otimes \vartheta_- - \vartheta_- \otimes \vartheta_+ \right) \quad \text{for all } \theta.$$

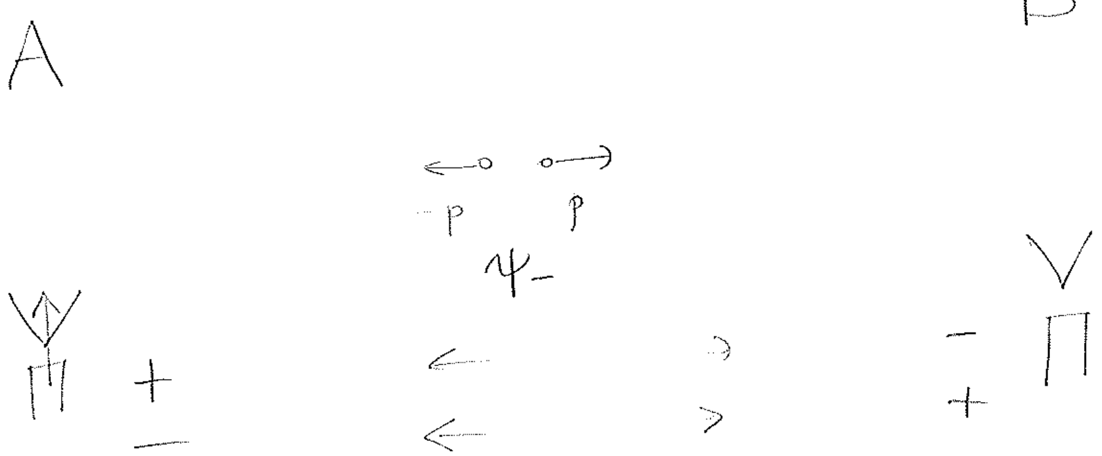
(Exercise: check this explicitly in the "lexicographic" rep of the tensor product)

EPR

(Einstein, Podolski, Rosen, 1935)

Recall that Stern-Gerlach splitting depends on the direction of the magnetic field. This direction determines the observable, e.g. σ_3; then the noncommuting observable σ_2 cannot be measured simultaneously. The simplest consistent —or, not obviously contradictive— interpretation of this is that the spin component in 3-direction has no value independent of a measurement that determines it.

EPR gave an argument that implies that local realism is problematic. (version of Bohm)



perfect anticorrelation (by angular momentum conservation); A and B may be arbitrarily far from one another.

But if A finds +, this projects to $|+-\rangle$ hence implies that B must find −.
[Yet they can be so far apart that not even a light signal can join them in the short measuring time ...]

→ something appears nonlocal.

EPR say: no action-at-a-distance
⟹ 3-component must have been real all the time, since it cannot just be produced instantaneously from far, far away.

Bohr says: no.

A cannot transmit a signal using this since his measurement outcome is an ideal random sequence of bits (±1) — after all, his DM $\rho_A = \frac{1}{2}\mathbb{1}$.

John Bell found a clever way to check assumptions about local & real theories.

Bell's inequality I

Let A measure along Θ_i and B along $\tilde{\Theta}_j$

P_{ij} = probability that A gets + and B −
or A gets − and B +

$$= \langle \psi_- | P_+(\Theta_i) \otimes P_-(\tilde{\Theta}_j) \psi_- \rangle$$
$$+ \langle \psi_- | P_-(\Theta_i) \otimes P_+(\tilde{\Theta}_j) \psi_- \rangle$$

By rotation invariance,

$$\langle \psi_- | P_+(\Theta_i) \otimes P_-(\tilde{\Theta}_j) \psi_- \rangle$$
$$= \langle \psi_- | \underbrace{P_+(\Theta_i - \tilde{\Theta}_j)}_{} \otimes P_-(0) \psi_- \rangle$$

$$\underset{\vartheta_{ij} = \Theta_i - \tilde{\Theta}_j}{=} \frac{1}{2} (1,0) P_+(\vartheta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \cos^2 \frac{\vartheta}{2}$$

$$P_+(\vartheta) = \begin{pmatrix} \cos\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2} \end{pmatrix} \left(\cos\frac{\vartheta}{2}, \sin\frac{\vartheta}{2}\right) = \begin{pmatrix} \cos^2\frac{\vartheta}{2} & \cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2} & \sin^2\frac{\vartheta}{2} \end{pmatrix}$$

$$P_-(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{so} \quad A \otimes P_-(0) \psi_- = \frac{1}{\sqrt{2}} A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus $P_{ij} = 2 \frac{1}{2} \cos^2 \frac{\vartheta}{2} = \cos^2 \frac{\vartheta}{2} = \frac{1}{2}(1 + \cos\vartheta)$

~~Let $\tilde{\Theta}_j = \Theta_j + \pi$ then~~

$$\cancel{P_{ij} = \frac{1}{2}(1 - \cos(\Theta_i - \Theta_j))}.$$

(18)

A measures at angles $\theta_1 = 0$, $\theta_2 = \frac{2\pi}{3}$, $\theta_3 = \frac{4\pi}{3}$

B measures at $\tilde{\theta}_j = \theta_j + \pi$

i.e. B gets the <u>same</u> answer as A when $i = j$.

$$\theta_i - \theta_j = \pm \frac{2\pi}{3} \mod 2\pi$$

$$\Rightarrow \cos(\theta_i - \theta_j) = \cos\frac{2\pi}{3} = -\frac{1}{2}$$

$$\Rightarrow P_{ij} = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

Thus $P_{12} + P_{23} + P_{13} = \frac{3}{4}$.

Now think of 3 coins that are thrown at each measurement, both by A and B but only one of them is looked at.

[$\rho \sim \mathbb{1}$, so it's a toss of an even coin for each of them separately]

Let $\tilde{P}_{ij}$ be the probability that A and B see the same side of the coin. Then

$$\tilde{P}_{12} + \tilde{P}_{13} + \tilde{P}_{23} \geq 1$$

since there are 3 coins and just 2 sides of a coin

so the statistics of the joint measurement results does not fit a probabilistic picture.

The problem is <u>not</u> a correlation, but an inequality from positivity of probabilities; and the fact that one can in principle not throw all 3 coins in a row.

Bell's inequality II

Bell's inequality is much more than this comparison. It rules out a large class of local hidden variable theories.

HVT: add'l variables that complete the QM description to a deterministic one, where particle trajectories are back and quantum expectations are ensemble averages.

von Neumann (1930s) $\nexists$ HVT (proof)

Bohm (1952): explicit construction of HVT's.

$$j_\psi = \frac{\hbar}{2m} \operatorname{Im}(\psi^* \nabla \psi) \qquad (\psi^* = \bar{\psi}^T)$$

$$\rho_\psi = \psi^* \psi$$

$$\lambda \in \mathbb{R}^3: \quad \frac{d\lambda}{dt} = \frac{j_\psi(t,\lambda)}{\rho_\psi(t,\lambda)}$$

ensemble (λ, ψ) λ: appear with prob $\rho(t,\lambda) = \rho_\psi(t,\lambda)$

(i.e. the "hidden" variables are position variables).

But: "nonlocal" "quantum potential".

Bell's Theorem (1964)

There are no local hidden variables.

Proof. Suppose $\exists$ prob. measure ρ $\left(\int d\rho(\lambda)=1\right)$ so that for all unit vectors $\hat{a}, \hat{b}$, the result of the measurement of

$$\vec{\sigma}\cdot\hat{a}\otimes\mathbb{1} = A(\hat{a},\lambda) = \pm 1$$

and the result of the measurement of

$$\mathbb{1}\otimes\vec{\sigma}\cdot\hat{b} = B(\hat{b},\lambda) = \pm 1$$

Here λ stands for an arbitrary number of continuous or discrete variables, which can be divided up among A and B in an arbitrary way. [Think, e.g., of initial values of Bohms hidden positions λ which depend on time.]

The essential assumptions are:

ρ a prob. measure (i.e. nonnegative)

$A(\hat{a},\lambda)$ is independent of $\hat{b}$

$B(\hat{b},\lambda)$ — " — $\hat{a}$

(no action at a distance).

To describe QM, ρ must satisfy

$$\int d\rho(\lambda)\, A(\hat{a},\lambda)\, B(\hat{b},\lambda) = \langle \vec{\sigma}\cdot\hat{a} \otimes \vec{\sigma}\cdot\hat{b} \rangle = -\vec{a}\cdot\vec{b}.$$

Precise form of Bell's Theorem:
There is no such ρ.

Proof. $R(a,b) := \int d\rho(\lambda)\, A(\hat{a},\lambda)\, B(\hat{b},\lambda)$.
fulfills $|R(a,b)| \leq 1$ since $A, B \in \pm 1$, ρ normalized.

<u>$a=b$</u>.

$$R(a,a) = \int d\rho(\lambda)\, A(\hat{a},\lambda)\, B(\hat{b},\lambda) = -1$$

$$\Leftrightarrow \quad A(\hat{a},\lambda) = -B(\hat{b},\lambda) \quad \rho\text{-a.e.}$$

$$\Rightarrow R(a,b) = -\int d\rho(\lambda)\, A(\hat{a},\lambda)\, A(\hat{b},\lambda).$$

Let $\hat{c}$ be another unit vector

$$R(a,b) - R(a,c) = -\int d\rho(\lambda) \left[A(a,\lambda)A(b,\lambda) - A(a,\lambda)A(c,\lambda)\right]$$

$$\underset{A(b,\lambda)^2=1}{=} -\int d\rho(\lambda)\, A(a,\lambda) A(b,\lambda) \underbrace{\left[1 - A(b,\lambda)A(c,\lambda)\right]}_{\geq 0}$$

$$\Rightarrow |R(a,b) - R(a,c)| \leq \int d\rho(\lambda) \left(1 - A(b,\lambda)A(c,\lambda)\right)$$

$$= 1 + R(b,c).$$

This is Bell's inequality:

$$\left| R(a,b) - R(a,c) \right| \leq 1 + R(b,c)$$

In our Stern-Gerlach example,

$$R(b,c) = -\cos(\theta_b - \theta_c),$$

$R(b,c)$ is minimal for $\theta_b - \theta_c = \theta_{bc} = 0$.

Choose $\theta_a = \frac{\pi}{2}$ and consider small θ_{bc}.

$$1 - \cos(\theta_{bc}) \geq \left| \cos(\underbrace{\theta_a}_{\pi/2} - \theta_b) - \cos(\underbrace{\theta_a}_{\pi/2} - \theta_c) \right|$$

$$= \left| \sin\theta_b - \sin\theta_c \right|$$

$$= 2 \left| \cos\frac{\theta_b + \theta_c}{2} \sin\frac{\theta_b - \theta_c}{2} \right|$$

$$\frac{\theta_{bc}^2}{2} \text{ "}\geq\text{" } 2\,\theta_{bc} \cos\frac{\theta_b + \theta_c}{2}$$

up to higher orders.

Now take $\theta_a = 0$, $\theta_b = \frac{\pi}{3}$, $\theta_c = \frac{2\pi}{3}$.

Then $R_{ab} = -\cos(\theta_a - \theta_b) = -\cos\frac{\pi}{3} = -\frac{1}{2}$,

$R_{ac} = -\cos(\theta_a - \theta_c) = -\cos\frac{2\pi}{3} = \frac{1}{2}$

$R_{bc} = -\cos(\theta_b - \theta_c) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

Thus $\left| R_{ab} - R_{ac} \right| = 1 \nleq 1 - \frac{1}{2} = 1 + R_{bc}$

$\boxed{\text{Bell's inequality is violated in QM}}$

Purification.

We have seen that reduction to a subsystem maps pure states on $\mathcal{H}_1 \otimes \mathcal{H}_2$ to mixed states on $\mathcal{H}_i$ (if there is any entanglement between 1 and 2). Purification is going into the opposite direction, simply by "doubling".

Let $\mathcal{H}$ be the system Hilbert space and ρ a DM on $\mathcal{H}$. Let

$$\rho = \sum_k p_k |\varphi_k\rangle\langle\varphi_k|$$

be the spectral representation of ρ ($\rho\varphi_k = p_k\varphi_k$, $\|\varphi_k\| = 1$)

Enlarge the Hilbert space to $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}$

and set

$$\psi = \sum_k \sqrt{p_k}\, \varphi_k \otimes \varphi_k$$

and $\tilde{\rho} = |\psi\rangle\langle\psi|$

(ψ = normalized since $\sum p_k = 1$)

Then $\rho_1 = \mathrm{Tr}_2 \tilde{\rho}$ has matrix elements

$$\langle\varphi_k|\rho_1\varphi_{k'}\rangle = \sum_{m\,m'} \sqrt{p_m p_{m'}} \sum_l \underbrace{\langle\varphi_k\otimes\varphi_l|\varphi_m\otimes\varphi_m\rangle\langle\varphi_{m'}\otimes\varphi_{m'}|\varphi_{k'}\otimes\varphi_l\rangle}_{\delta_{km}\,\delta_{lm}\,\delta_{m'k'}\,\delta_{m'l}}$$

$$= \delta_{kk'}\, p_k$$

so ρ is the reduced DM of the pure state $\tilde{\rho}$.

Relative entropy.

Classically, the relative entropy of two probability distributions is

$$H(p|q) = \sum_i p_i \log \frac{p_i}{q_i}$$

it satisfies $H(p|q) \geq 0$.

A similar inequality holds for von Neumann entropies. Let ρ and ρ' be DM. Define the relative entropy as

$$S(\rho|\rho') = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \rho')$$

Theorem: $S(\rho|\rho') \geq 0$.

This follows from Klein's inequality:

Lemma. If F and G are positive operators with $\mathrm{Tr}\, F < \infty$ and $\mathrm{Tr}\, G < \infty$, then

$$0 \leq \mathrm{Tr}\left[F(\ln F - \ln G) - (F - G)\right]$$

and "=" holds iff $F = G$.

[NB: in general, $[F, G] \neq 0$!]

Proof.

$F = F^* \geq 0 \implies F\phi_n = f_n \phi_n, \; f_n \geq 0 \; (\phi_n)_n$ ONB

$G = G^* \geq 0 \implies G\psi_n = g_n \psi_n, \; g_n \geq 0 \; (\psi_n)_n$ ONB

We may assume that $G > 0$ (by replacing $G \to G_\varepsilon$, with EV $g_n + \varepsilon$, and taking $\varepsilon \to 0$).

We may also assume $f_n > 0$ because the $f_n = 0$ terms won't contribute to the following expressions.

$$\mathrm{Tr}[F(\ln F - \ln G)]$$

$$= \sum_m \langle \phi_m | F(\ln F - \ln G)\phi_m \rangle$$

$$= \sum_m f_m \langle \phi_m | (\ln f_m \mathbb{1} - \ln G)\phi_m \rangle$$

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n|$$

$$= \sum_{m,n} f_m \underbrace{(\ln f_m - \ln g_n)}_{\ln \frac{f_m}{g_n} = -\ln \frac{g_n}{f_m} \geq -\left(\frac{g_m}{f_n} - 1\right)} |\langle \phi_m | \psi_n \rangle|^2$$

$$\ln(1+x) \leq x$$

$$\geq -\sum_{m,n} (g_n - f_m) \langle \phi_m | \psi_n \rangle \langle \psi_n | \phi_m \rangle$$

$$= -\sum_{m,n} \langle \phi_m | (G - F)\psi_n \rangle \langle \psi_n | \phi_m \rangle =$$

$$= -\sum_m \langle \phi_m | G - F \phi_m \rangle = \mathrm{Tr}(F - G).$$

$\blacksquare$

Applications

(1) Let ρ be a DM on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and

$\mathrm{Tr}(\rho(A_1 \otimes \mathbb{1})) = \mathrm{Tr}(\rho_1 A_1)$ and $\mathrm{Tr}(\rho(\mathbb{1} \otimes A_2)) = \mathrm{Tr}(\rho_2 A_2)$.

Then

$$S(\rho \,|\, \rho_1 \otimes \rho_2) =$$

full state

state where correlations dropped;

$$= \mathrm{Tr}(\rho \ln \rho) - \mathrm{Tr}(\rho \ln(\rho_1 \otimes \rho_2))$$

$$= -S(\rho) - \mathrm{Tr}(\rho(\ln(\rho_1 \otimes \mathbb{1}) + \ln(\mathbb{1} \otimes \rho_2)))$$

these operators commute!

$$= -S(\rho) - \mathrm{Tr}(\rho \ln(\rho_1 \otimes \mathbb{1})) - \mathrm{Tr}(\rho \ln(\mathbb{1} \otimes \rho_2))$$

$$= -S(\rho) - \mathrm{Tr}(\rho_1 \ln \rho_1) - \mathrm{Tr}(\rho_2 \ln \rho_2)$$

$$= S(\rho_1) + S(\rho_2) - S(\rho)$$

$$\geq 0$$

$$\Rightarrow S(\rho) \leq S(\rho_1) + S(\rho_2)$$

"information loss upon partial tracing"

2. Entropy increase under von Neumann measurement. ($\dim \mathcal{H} < \infty$)

$A = A^*$ observable, $A = \sum_i a_i P_i$ ← proj to eigenspace of a_i

Probability of finding a_i in state ρ is

$$w_i = tr(P_i \rho)$$

After measuring a_i the state is

$$\frac{1}{w_i} P_i \rho P_i = \frac{1}{w} P_i \rho P_i^* \qquad \text{since } P_i = P_i^*$$

Thus after the measurement, irrespective of the outcome, have

$$\rho' = \sum_i w_i \frac{1}{w_i} P_i \rho P_i^* = \sum_i P_i \rho P_i^*$$

Claim: $S(\rho') \geq S(\rho)$.

Proof.

$$0 \leq S(\rho | \rho') = Tr(\rho \ln \rho) - Tr(\rho \ln \rho')$$

$$-Tr(\rho \ln \rho') \underset{\sum P_i = 1}{=} -\sum_i Tr(P_i \rho \ln \rho') \underset{\substack{P_i^2 = P_i \\ + \text{cyclicity}}}{=} -\sum_i Tr(P_i \rho \ln \rho' P_i)$$

$$\underset{\substack{P_i \rho' = \rho' P_i \\ \text{since } P_i P_j = 0,\, i \neq j}}{=} -\sum_i Tr(P_i \rho P_i \ln \rho') = -Tr(\rho' \ln \rho')$$

Thus $0 \leq S(\rho | \rho') = S(\rho') - S(\rho)$. ∎