C Magnetostatics

In this chapter we consider magnetostatics starting from the equations, which were derived at the beginning of section (3.a) for time independent currents.

9 Magnetic Induction and Vector Potential

9.a Ampere's Law

From

$$\operatorname{curl}\mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r}) \tag{9.1}$$

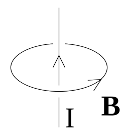
one obtains

$$\int \mathrm{d}\mathbf{f}\cdot \operatorname{curl}\mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}\int \mathrm{d}\mathbf{f}\cdot\mathbf{j}(\mathbf{r}), \tag{9.2}$$

which can be written by means of Stokes' theorem (B.56)

$$\oint \mathrm{d}\mathbf{r}\cdot\mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}I. \tag{9.3}$$

The line integral of the magnetic induction **B** along a closed line yields $4\pi/c$ times the current I through the line.Here the corkscrew rule applies: If the current moves in the direction of the corkscrew, then the magnetic induction has the direction in which the corkscrew rotates.



9.b Magnetic Flux

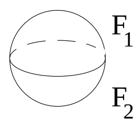
The magnetic flux Ψ^{m} through an oriented area F is defined as the integral

$$\Psi^{\mathrm{m}} = \int_F \mathrm{d}\mathbf{f}\cdot\mathbf{B}(\mathbf{r}). \tag{9.4}$$

The magnetic flux depends only on the boundary ∂F of the area. To show this we consider the difference of the flux through two areas F_1 and F_2 with the same boundary and obtain

$$\Psi^{\mathrm{m}}_1 - \Psi^{\mathrm{m}}_2 = \int_{F_1}\mathrm{d}\mathbf{f}\cdot\mathbf{B}(\mathbf{r}) - \int_{F_2}\mathrm{d}\mathbf{f}\cdot\mathbf{B}(\mathbf{r}) = \int_F \mathrm{d}\mathbf{f}\cdot\mathbf{B}(\mathbf{r}) = \int_V \mathrm{d}^3r \operatorname{div}\mathbf{B}(\mathbf{r}) = 0 \tag{9.5}$$

by means of the divergence theorem (B.59) and $\operatorname{div}\mathbf{B}(\mathbf{r}) = 0$.Suppose F_1 and F_2 are oriented in the same direction (for example upwards). Then the closed surface F is composed of F_1 and F_2, where F_2 is now oriented in the opposite direction.Then F has a definite orientation (for example outwards) and includes the volume V.



9.c Field of a Current Distribution

From curl curl $\mathbf{B}(\mathbf{r}) = (4\pi/c)\,\mathrm{curl}\,\mathbf{j}(\mathbf{r})$ due to (B.26)

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{B}(\mathbf{r}) = \mathrm{grad}\,\mathrm{div}\,\mathbf{B}(\mathbf{r}) - \triangle\mathbf{B}(\mathbf{r}) \tag{9.6}$$

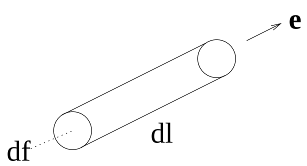
and $\mathrm{div}\,\mathbf{B}(\mathbf{r}) = 0$ one obtains

$$\triangle\mathbf{B}(\mathbf{r}) = -\frac{4\pi}{c}\,\mathrm{curl}\,\mathbf{j}(\mathbf{r}) \tag{9.7}$$

with the solution

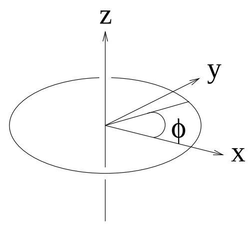
$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\int \mathrm{d}^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{curl}'\,\mathbf{j}(\mathbf{r}') = -\frac{1}{c}\int \mathrm{d}^3r' \Big(\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big) \times \mathbf{j}(\mathbf{r}') = \frac{1}{c}\int \mathrm{d}^3r' \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3} \times \mathbf{j}(\mathbf{r}'), \tag{9.8}$$

where we have used (B.63) at the second equals sign. The last expression is called the law of Biot and Savart.If the extension of a wire perpendicular to the direction of the current is negligible (filamentary wire) then one can approximate $\mathrm{d}^3r'\mathbf{j}(\mathbf{r}') = \mathrm{d}f'\mathrm{d}l'\,j(\mathbf{r}')\mathbf{e} = I\mathrm{d}\mathbf{r}'$ and obtains



$$\mathbf{B}(\mathbf{r}) = \frac{I}{c}\int \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3} \times \mathrm{d}\mathbf{r}' \tag{9.9}$$

As an example we consider the induction in the middle axis of a current along a circle



$$\mathbf{r} = z\mathbf{e}_z, \quad \mathbf{r}' = (R\cos\phi, R\sin\phi, z') \quad \mathrm{d}\mathbf{r}' = (-R\sin\phi, R\cos\phi, 0)\mathrm{d}\phi \tag{9.10}$$

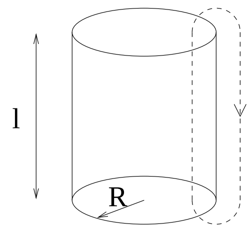
$$(\mathbf{r}'-\mathbf{r}) \times \mathrm{d}\mathbf{r}' = (R(z-z')\cos\phi, R(z-z')\sin\phi, R^2)\mathrm{d}\phi \tag{9.11}$$

$$\mathbf{B}(0,0,z) = \frac{2\pi I R^2 \mathbf{e}_z}{c(R^2+(z-z')^2)^{3/2}}. \tag{9.12}$$

Starting from this result we calculate the field in the axis of a coil. The number of windings be N and it extends from $z' = -l/2$ to $z' = +l/2$. Then we obtain

$$\mathbf{B}(0,0,z) = \int_{-l/2}^{+l/2} \frac{N\mathrm{d}z'}{l} \frac{2\pi I R^2 \mathbf{e}_z}{c(R^2+(z-z')^2)^{3/2}} = \frac{2\pi I N}{cl}\mathbf{e}_z \left(\frac{\frac{l}{2}-z}{\sqrt{R^2+(\frac{l}{2}-z)^2}} + \frac{\frac{l}{2}+z}{\sqrt{R^2+(\frac{l}{2}+z)^2}} \right). \tag{9.13}$$

If the coil is long, $R \ll l$, then one may neglect R^2 and obtains inside the coil



$$\mathbf{B} = \frac{4\pi I N}{cl}\mathbf{e}_z. \tag{9.14}$$

At the ends of the coil the field has decayed to one half of its intensity inside the coil. From Ampere's law one obtains by integration along the path described in the figure

$$\oint \mathrm{d}\mathbf{r}\cdot\mathbf{B} = \frac{4\pi}{c} I N. \tag{9.15}$$

Thus inside the coil one obtains the induction (9.14), whereas the magnetic induction outside is comparatively small.

9.d Vector Potential

We now rewrite the expression for the magnetic induction

$$\mathbf{B}(\mathbf{r}) = -\frac{1}{c}\int \mathrm{d}^3r' \Big(\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big) \times \mathbf{j}(\mathbf{r}') = \frac{1}{c}\int \mathrm{d}^3r' \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \times \mathbf{j}(\mathbf{r}') = \mathrm{curl}\,\mathbf{A}(\mathbf{r}) \tag{9.16}$$

with

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathrm{d}^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{9.17}$$

One calls **A** the vector potential. Consider the analog relation between charge density ρ and the electric potential ϕ in electrostatics (3.14). We show that **A** is divergence-free

$$\begin{aligned}\operatorname{div}\mathbf{A}(\mathbf{r}) &= \frac{1}{c}\int \mathrm{d}^3r' \left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{j}(\mathbf{r}') = -\frac{1}{c}\int \mathrm{d}^3r' \left(\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{j}(\mathbf{r}') \\ &= \frac{1}{c}\int \mathrm{d}^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\mathbf{j}(\mathbf{r}') = 0.\end{aligned} \tag{9.18}$$

At the third equals sign we have performed a partial integration (B.62). Finally we have used $\operatorname{div}\mathbf{j}(\mathbf{r}) = 0$.

9.e Force Between Two Circuits

Finally, we consider the force between two circuits. The force exerted by circuit (1) on circuit (2) is

$$\begin{aligned}\mathbf{K}_2 &= \frac{1}{c}\int \mathrm{d}^3r\,\mathbf{j}_2(\mathbf{r})\times\mathbf{B}_1(\mathbf{r}) = \frac{1}{c^2}\int \mathrm{d}^3r\mathrm{d}^3r'\,\mathbf{j}_2(\mathbf{r})\times\left(\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{j}_1(\mathbf{r}')\right) \\ &= \frac{1}{c^2}\int \mathrm{d}^3r\mathrm{d}^3r'\big(\mathbf{j}_1(\mathbf{r}')\cdot\mathbf{j}_2(\mathbf{r})\big)\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{c^2}\int \mathrm{d}^3r\mathrm{d}^3r'\left(\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{j}_2(\mathbf{r})\right)\mathbf{j}_1(\mathbf{r}')\end{aligned} \tag{9.19}$$

where (B.14) has been applied. Since due to (B.62)

$$\int \mathrm{d}^3r\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{j}_2(\mathbf{r}) = -\int \mathrm{d}^3r\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla\cdot\mathbf{j}_2(\mathbf{r}) \tag{9.20}$$

and $\operatorname{div}\mathbf{j}_2(\mathbf{r}) = 0$, one obtains finally for the force

$$\mathbf{K}_2 = \frac{1}{c^2}\int \mathrm{d}^3r\mathrm{d}^3r'\big(\mathbf{j}_1(\mathbf{r}')\cdot\mathbf{j}_2(\mathbf{r})\big)\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{9.21}$$

The force acting on circuit (1) is obtained by exchanging 1 and 2. Simultaneously, one can exchange **r** and **r**′. One sees then that

$$\mathbf{K}_1 = -\mathbf{K}_2 \tag{9.22}$$

holds.

Exercise Calculate the force between two wires of length l carrying currents I_1 and I_2 which run parallel in a distance r ($r \ll l$). Kohlrausch and Weber measured this force in order to determine the velocity of light.

10 Loops of Current as Magnetic Dipoles

10.a Localized Current Distribution and Magnetic Dipole

We consider a distribution of currents which vanishes outside a sphere of radius R ($\mathbf{j}(\mathbf{r}') = \mathbf{0}$ for $r' > R$) and determine the magnetic induction $\mathbf{B}(\mathbf{r})$ for $r > R$. We may expand the vector potential $\mathbf{A}(\mathbf{r})$ (9.17) similar to the electric potential $\Phi(\mathbf{r})$ in section (4)

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{cr}\int d^3r' \mathbf{j}(\mathbf{r}') + \frac{x_\alpha}{cr^3}\int d^3r' x'_\alpha \mathbf{j}(\mathbf{r}') + ... \tag{10.1}$$

Since no current flows through the surface of the sphere one obtains

$$0 = \int d\mathbf{f}\cdot g(\mathbf{r})\mathbf{j}(\mathbf{r}) = \int d^3r \operatorname{div}(g(\mathbf{r})\mathbf{j}(\mathbf{r})) = \int d^3r \operatorname{grad} g(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}) + \int d^3r g(\mathbf{r}) \operatorname{div}\mathbf{j}(\mathbf{r}), \tag{10.2}$$

where the integrals are extended over the surface and the volume of the sphere, respectively. From the equation of continuity (1.12,3.1) it follows that

$$\int d^3r \operatorname{grad} g(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}) = 0. \tag{10.3}$$

This is used to simplify the integral in the expansion (10.1). With $g(\mathbf{r}) = x_\alpha$ one obtains

$$\int d^3r\, j_\alpha(\mathbf{r}) = 0. \tag{10.4}$$

Thus the first term in the expansion vanishes. There is no contribution to the vector potential decaying like $1/r$ in magnetostatics, i.e. there is no magnetic monopole. With $g(\mathbf{r}) = x_\alpha x_\beta$ one obtains

$$\int d^3r\Big(x_\alpha j_\beta(\mathbf{r}) + x_\beta j_\alpha(\mathbf{r})\Big) = 0. \tag{10.5}$$

Thus we can rewrite

$$\int d^3r x_\alpha j_\beta = \frac{1}{2}\int d^3r(x_\alpha j_\beta - x_\beta j_\alpha) + \frac{1}{2}\int d^3r(x_\alpha j_\beta + x_\beta j_\alpha). \tag{10.6}$$

The second integral vanishes, as we have seen. The first one changes its sign upon exchanging the indices α and β. One introduces

$$\int d^3r x_\alpha j_\beta = \frac{1}{2}\int d^3r(x_\alpha j_\beta - x_\beta j_\alpha) = c\epsilon_{\alpha,\beta,\gamma} m_\gamma \tag{10.7}$$

and calls the resulting vector

$$\mathbf{m} = \frac{1}{2c}\int d^3r'(\mathbf{r}'\times\mathbf{j}(\mathbf{r}')) \tag{10.8}$$

magnetic dipole moment.. Then one obtains

$$A_\beta(\mathbf{r}) = \frac{x_\alpha}{cr^3} c\epsilon_{\alpha,\beta,\gamma} m_\gamma + ... \tag{10.9}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m}\times\mathbf{r}}{r^3} + ... \tag{10.10}$$

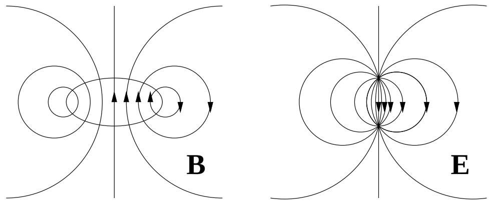
with $\mathbf{B}(\mathbf{r}) = \operatorname{curl}\mathbf{A}(\mathbf{r})$ one obtains

$$\mathbf{B}(\mathbf{r}) = \frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r}) - \mathbf{m}r^2}{r^5} + ... \tag{10.11}$$

This is the field of a magnetic dipole. It has the same form as the electric field of an electric dipole (4.12)

$$\mathbf{E}(\mathbf{r}) = -\operatorname{grad}\Big(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\Big) = \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r}) - \mathbf{p}r^2}{r^5}, \tag{10.12}$$

but there is a difference at the location of the dipole. This can be seen in the accompanying figure. Calculate the $\delta^3(\mathbf{r})$-contribution to both dipolar moments. Compare (B.71).



10.b Magnetic Dipolar Moment of a Current Loop

The magnetic dipolar moment of a current on a closed curve yields

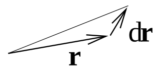
$$\mathbf{m} = \frac{I}{2c}\int \mathbf{r}\times \mathrm{d}\mathbf{r} = \frac{I}{c}\mathbf{f}, \tag{10.13}$$

e.g.

$$m_z = \frac{I}{2c}\int (x\mathrm{d}y - y\mathrm{d}x) = \frac{I}{c}f_z. \tag{10.14}$$

Here f_α is the projection of the area inside the loop onto the plane spanned by the two other axes

$$\mathrm{d}\mathbf{f} = \frac{1}{2}\mathbf{r}\times \mathrm{d}\mathbf{r}. \tag{10.15}$$



If $\mathbf{j} = \sum_i q_i \mathbf{v}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$, then using (10.8) the magnetic moment reads

$$\mathbf{m} = \frac{1}{2c}\sum_i q_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i \frac{q_i}{2m_i c}\mathbf{l}_i, \tag{10.16}$$

where m_i is the mass and $\mathbf{l}_i$ the angular momentum. If only one kind of charges is dealt with, then one has

$$\mathbf{m} = \frac{q}{2mc}\mathbf{l}. \tag{10.17}$$

This applies for orbital currents. For spins, however, one has

$$\mathbf{m} = \frac{q}{2mc}g\mathbf{s}, \tag{10.18}$$

where $\mathbf{s}$ is the angular momentum of the spin. The gyromagnetic factor for electrons is $g = 2.0023$ and the components of the spin $\mathbf{s}$ are $\pm\hbar/2$. Since in quantum mechanics the orbital angular momentum assumes integer multiples of $\hbar$, one introduces as unit for the magnetic moment of the electron Bohr's magneton, $\mu_B = \frac{e_0\hbar}{2m_0 c} = 0.927\cdot 10^{-20}\ \mathrm{dyn}^{1/2}\ \mathrm{cm}^2$.

10.c Force and Torque on a Dipole in an External Magnetic Field

10.c.α Force

An external magnetic induction $\mathbf{B}_\mathrm{a}$ exerts on a loop of a current the Lorentz force

$$\mathbf{K} = \frac{1}{c}\int \mathrm{d}^3r\mathbf{j}(\mathbf{r})\times\mathbf{B}_\mathrm{a}(\mathbf{r}) = -\frac{1}{c}\mathbf{B}_\mathrm{a}(0)\times\int \mathrm{d}^3r\mathbf{j}(\mathbf{r}) - \frac{1}{c}\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}\times\int \mathrm{d}^3r x_\alpha j_\beta(\mathbf{r})\mathbf{e}_\beta - ... = -\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}\times\mathbf{e}_\beta\epsilon_{\alpha,\beta,\gamma}m_\gamma. \tag{10.19}$$

We rewrite $m_\gamma\epsilon_{\alpha,\beta,\gamma}\mathbf{e}_\beta = m_\gamma\mathbf{e}_\gamma\times\mathbf{e}_\alpha = \mathbf{m}\times\mathbf{e}_\alpha$ and find

$$\mathbf{K} = -\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}\times(\mathbf{m}\times\mathbf{e}_\alpha) = (\mathbf{m}\cdot\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha})\mathbf{e}_\alpha - (\mathbf{e}_\alpha\cdot\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha})\mathbf{m}. \tag{10.20}$$

The last term vanishes because of $\operatorname{div}\mathbf{B} = 0$. The first term on the right hand side can be written $(\mathbf{m}\cdot\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha})\mathbf{e}_\alpha = m_\gamma \frac{\partial B_{\mathrm{a},\gamma}}{\partial x_\alpha}\mathbf{e}_\alpha = m_\gamma \frac{\partial B_{\mathrm{a},\alpha}}{\partial x_\gamma}\mathbf{e}_\alpha = (\mathbf{m}\nabla)\mathbf{B}_\mathrm{a}$, where we have used $\operatorname{curl}\mathbf{B}_\mathrm{a} = \mathbf{0}$ in the region of the dipole. Thus we obtain the force

$$\mathbf{K} = (\mathbf{m}\,\mathrm{grad}\,)\mathbf{B}_\mathrm{a} \tag{10.21}$$

acting on the magnetic dipole expressed by the vector gradient (B.18). This is in analogy to (4.35), where we obtained the force $(\mathbf{p}\,\mathrm{grad}\,)\mathbf{E}_\mathrm{a}$ acting on an electric dipole.

10.c.β Torque

The torque on a magnetic dipole is given by

$$\mathbf{M}_\mathrm{mech} = \frac{1}{c}\int \mathrm{d}^3 r\,\mathbf{r}\times(\mathbf{j}\times\mathbf{B}_\mathrm{a}) = -\frac{1}{c}\mathbf{B}_\mathrm{a}\int \mathrm{d}^3 r(\mathbf{r}\cdot\mathbf{j}) + \frac{1}{c}\int \mathrm{d}^3 r(\mathbf{B}_\mathrm{a}\cdot\mathbf{r})\mathbf{j}. \tag{10.22}$$

The first integral vanishes, which is easily seen from (10.3) and $g = r^2/2$. The second integral yields

$$\mathbf{M}_\mathrm{mech} = \frac{1}{c}\mathbf{e}_\beta B_{\mathrm{a},\alpha}\int \mathrm{d}^3 r\, x_\alpha j_\beta = B_{\mathrm{a},\alpha}\mathbf{e}_\beta\epsilon_{\alpha,\beta,\gamma}m_\gamma = \mathbf{m}\times\mathbf{B}_\mathrm{a} \tag{10.23}$$

Analogously the torque on an electric dipole was $\mathbf{p}\times\mathbf{E}_\mathrm{a}$, (4.36).
From the law of force one concludes the energy of a magnetic dipole in an external field as

$$U = -\mathbf{m}\cdot\mathbf{B}_\mathrm{a}. \tag{10.24}$$

This is correct for permanent magnetic moments. However, the precise derivation of this expression becomes clear only when we treat the law of induction (section 13).

11 Magnetism in Matter. Field of a Coil

11.a Magnetism in Matter

In a similar way as we separated the polarization charges from freely accessible charges, we divide the current density into a freely moving current density $\mathbf{j}_\mathrm{f}$ and the density of the magnetization current $\mathbf{j}_\mathrm{M}$, which may come from orbital currents of electrons

$$\mathbf{j}(\mathbf{r}) = \mathbf{j}_\mathrm{f}(\mathbf{r}) + \mathbf{j}_\mathrm{M}(\mathbf{r}). \tag{11.1}$$

We introduce the magnetization as the density of magnetic dipoles

$$\mathbf{M} = \frac{\sum \mathbf{m}_i}{\Delta V} \tag{11.2}$$

and conduct the continuum limit

$$\sum_i \mathbf{m}_i f(\mathbf{r}_i) \to \int \mathrm{d}^3r' \mathbf{M}(\mathbf{r}') f(\mathbf{r}'). \tag{11.3}$$

Then using (10.10) we obtain for the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathrm{d}^3r' \frac{\mathbf{j}_\mathrm{f}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int \mathrm{d}^3r' \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{11.4}$$

The second integral can be rewritten

$$\int \mathrm{d}^3r' \mathbf{M}(\mathbf{r}') \times \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \int \mathrm{d}^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \times \mathbf{M}(\mathbf{r}'), \tag{11.5}$$

so that one obtains

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathrm{d}^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\mathbf{j}_\mathrm{f}(\mathbf{r}') + c\,\mathrm{curl}'\mathbf{M}(\mathbf{r}')). \tag{11.6}$$

Thus one interprets

$$\mathbf{j}_\mathrm{M}(\mathbf{r}') = c\,\mathrm{curl}'\mathbf{M}(\mathbf{r}') \tag{11.7}$$

as the density of the magnetization current. Then one obtains for the magnetic induction

$$\mathrm{curl}\,\mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}_\mathrm{f}(\mathbf{r}) + 4\pi\,\mathrm{curl}\,\mathbf{M}(\mathbf{r}). \tag{11.8}$$

Now one introduces the magnetic field strength

$$\mathbf{H}(\mathbf{r}) := \mathbf{B}(\mathbf{r}) - 4\pi\mathbf{M}(\mathbf{r}) \tag{11.9}$$

for which MAXWELL's equation

$$\mathrm{curl}\,\mathbf{H}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}_\mathrm{f}(\mathbf{r}) \tag{11.10}$$

holds. MAXWELL's equation $\mathrm{div}\,\mathbf{B}(\mathbf{r}) = 0$ remains unchanged.
One obtains for paramagnetic and diamagnetic materials in not too strong fields

$$\mathbf{M} = \chi_\mathrm{m}\mathbf{H}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \mu = 1 + 4\pi\chi_\mathrm{m}, \tag{11.11}$$

where χ_m is the magnetic susceptibility and μ the permeability. In superconductors (of first kind) one finds complete diamagnetism $\mathbf{B} = \mathbf{0}$. There the magnetic induction is completely expelled from the interior by surface currents.
In analogy to the arguments for the dielectric displacement and the electric field one obtains that the normal component B_n is continuous, and in the absence of conductive currents also the tangential components $\mathbf{H}_\mathrm{t}$ are continuous across the boundary.
In the GAUSSian system of units the fields $\mathbf{M}$ and $\mathbf{H}$ are measured just as $\mathbf{B}$ in $\mathrm{dyn}^{1/2}\,\mathrm{cm}^{-1}$, whereas in SI-units $\mathbf{B}$ is measured in $\mathrm{Vs/m^2}$, $\mathbf{H}$ and $\mathbf{M}$ in A/m. The conversion factors for $\mathbf{H}$ and $\mathbf{M}$ differ by a factor 4π. For more information see appendix A.

11.b Field of a coil

The field of a coil along its axis was determined in (9.13). We will now determine the field of a cylindrical coil in general. In order to do so we firstly consider an electric analogy. The field between two charges q and $-q$ at $\mathbf{r}_2$ and $\mathbf{r}_1$ is equivalent to a line of electric dipoles $\mathrm{d}\mathbf{p} = q\mathrm{d}\mathbf{r}'$ from $\mathbf{r}' = \mathbf{r}_1$ to $\mathbf{r}' = \mathbf{r}_2$. Indeed we obtain for the potential

$$\Phi(\mathbf{r}) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{p}}{|\mathbf{r}-\mathbf{r}'|^3} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{q(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\frac{q}{2}\int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{\mathrm{d}(\mathbf{r}-\mathbf{r}')^2}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{q}{|\mathbf{r}-\mathbf{r}_2|} - \frac{q}{|\mathbf{r}-\mathbf{r}_1|} \tag{11.12}$$

and thus for the field

$$\mathbf{E}(\mathbf{r}) = q\left(\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3} - \frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3}\right). \tag{11.13}$$

The magnetic analogy is to think of a long thin coil as consisting of magnetic dipoles

$$\mathrm{d}\mathbf{m} = \frac{\mathrm{d}I}{\mathrm{d}l}\frac{f}{c}\mathrm{d}\mathbf{r} = \frac{NIf}{lc}\mathrm{d}\mathbf{r}. \tag{11.14}$$

If we consider that the field of the electric and the magnetic dipole have the same form (10.11, 10.12) except at the point of the dipole, then it follows that we may replace q by $q_\mathrm{m} = NIf/(lc)$ in order to obtain the magnetic induction

$$\mathbf{B}(\mathbf{r}) = q_\mathrm{m}\left(\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3} - \frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3}\right). \tag{11.15}$$

Thus the field has a form which can be described by two magnetic monopoles with strengths q_m und $-q_\mathrm{m}$. However, at the positions of the dipoles the field differs in the magnetic case. There, i.e. inside the coil an additional field $B = 4\pi NI/(lc)$ flows back so that the field is divergency free and fulfills Ampere's law.
In order to obtain the result in a more precise way one uses the following consideration: We represent the current density in analogy to (11.7) as curl of a fictitious magnetization $\mathbf{j}_\mathrm{f} = c\,\mathrm{curl}\,\mathbf{M}_\mathrm{f}(\mathbf{r})$ inside the coil, outside $\mathbf{M}_\mathrm{f} = \mathbf{0}$. For a cylindrical (its cross-section needs not be circular) coil parallel to the z-axis one puts simply $\mathbf{M}_\mathrm{f} = NI\mathbf{e}_z/(cl)$. Then one obtains from

$$\mathrm{curl}\,\mathbf{B} = \frac{4\pi}{c}\mathbf{j}_\mathrm{f}(\mathbf{r}) = 4\pi\,\mathrm{curl}\,\mathbf{M}_\mathrm{f} \tag{11.16}$$

the induction $\mathbf{B}$ in the form

$$\mathbf{B}(\mathbf{r}) = 4\pi\mathbf{M}_\mathrm{f}(\mathbf{r}) - \mathrm{grad}\,\Psi(\mathbf{r}). \tag{11.17}$$

The function Ψ is determined from

$$\mathrm{div}\,\mathbf{B} = 4\pi\,\mathrm{div}\,\mathbf{M}_\mathrm{f} - \triangle\Psi = 0 \tag{11.18}$$

as

$$\Psi(\mathbf{r}) = -\int \mathrm{d}^3r'\,\frac{\mathrm{div}'\mathbf{M}_\mathrm{f}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{11.19}$$

The divergency yields in the present case of a cylindrical coil a contribution $\delta(z-z_1)NI/(cl)$ at the covering and a contribution $-\delta(z-z_2)NI/(cl)$ at the basal surface of the coil, since the component of $\mathbf{B}$ normal to the surface jumps by $NI/(cl)$, which yields

$$\Psi(\mathbf{r}) = \frac{NI}{cl}\left(\int_{F_2}\frac{\mathrm{d}^2r'}{|\mathbf{r}-\mathbf{r}'|} - \int_{F_1}\frac{\mathrm{d}^2r'}{|\mathbf{r}-\mathbf{r}'|}\right), \tag{11.20}$$

where F_2 is the covering and F_1 the basal surface. Thus one obtains an induction, as if there were magnetic charge densities $\pm NI/(cl)$ per area at the covering and the basal surface. This contribution yields a discontinuity of the induction at these parts of the surface which is compensated by the additional contribution $4\pi\mathbf{M}_\mathrm{f}$ inside the coil. The total strength of pole $\pm q_\mathrm{m}$ is the area of the basal (ground) surface times the charge density per area. One calls $\Psi(\mathbf{r})$ the magnetic potential. In view of the additional contribution $4\pi\mathbf{M}_\mathrm{f}(\mathbf{r})$ in (11.17) it is in contrast to the potentials $\Phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ only of limited use. We will not use it in the following.
Exercise Calculate magnetic field and magnetic induction for the coil filled by a core of permeability μ.
Exercise Show that the z-component of the magnetic induction is proportional to the difference of the solid angles under which the (transparently thought) coil appears from outside and from inside.